Linear Matrix Inequalities in System Analysis

Chung-Yao Kao

Optimization and Systems Theory

Royal Institute of Technology

Sweden

Semi-Definite Programs

Semi-Definite Programs (SDP)

$$\begin{aligned} \text{minimize} \quad & c'x \\ \text{subj. to} \quad & F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \leq 0 \\ & Ax = b, \end{aligned}$$

where $F_i = F_i' \in R^{p \times p}$.

- convex optimization with generalized inequalities
- equivalent to a set of polynomial inequalities
- multiple LMIs can be combined into one
- many nonlinear convex problems can be cast as SDP
- can be efficiently solved (in polynomial time)

Proposition 1 (Schur Complement). *Let Q and R be symmetric matrices. The the following are equivalent.*

1. $\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \leq 0.$
2. $R \geq 0,\ Q - SR^{\dagger}S^T \geq 0,\ S(I - RR^{\dagger}) = 0.$

where $R^{\dagger}$ denote the pseudo inverse of R.

reminder: Let R have the singular value decomposition

$$R = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} U'$$

where U is orthogonal and Σ is diagonal. Then

$$R^{\dagger} = U \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} U'$$

Proof. The inequality in 1 holds if and only if

$$\begin{bmatrix} I & 0 \\ 0 & U' \end{bmatrix} \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} Q & S_1 & S_2 \\ S_1' & \Sigma & 0 \\ S_2' & 0 & 0 \end{bmatrix} \geq 0 \qquad (1)$$

where $\begin{bmatrix} S_1 & S_2 \end{bmatrix} = SU$ is appropriately partitioned. For (1) to hold, we need $S_2 = 0$ which is equivalent to $S(I - RR^{\dagger}) = 0$. Furthermore, the inequality

$$0 \leq \begin{bmatrix} Q & S_1 \\ S_1' & \Sigma \end{bmatrix} = \begin{bmatrix} I & S_1\Sigma^{-\frac{1}{2}} \\ 0 & \Sigma^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} Q - S_1\Sigma^{-1}S_1' & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ \Sigma^{-\frac{1}{2}}S_1' & \Sigma^{\frac{1}{2}} \end{bmatrix}$$

must hold. This implies that $Q - S_1\Sigma^{-1}S_1' \geq 0$ or equivalent $Q - S_1R^{\dagger}S_1' \geq 0$ □

Corollary 1 (Strict Schur Complement). *Let Q and R be symmetric matrices. The the following are equivalent.*

1. $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0.$

2. $R > 0$, $Q - SR^{-1}S^T > 0$.

Some Examples

1. QCQP as SDP: the Quadratically Constrained Quadratic Program (QCQP)

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subj. to} \quad & f_i(x) \leq 0, \ i = 1, \cdots, L \end{aligned}$$

where $f_i(x) = (Ax + b)'(Ax + b) - (c'x + d)$. can expressed as SDP (in x and t)

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subj. to} \quad & F_0(x, t) \geq 0 \\ & F_i(x) \geq 0, \ i = 1, \cdots, L \end{aligned}$$

where

$$F_0(x, t) = \begin{bmatrix} I & A_0x + b_0 \\ (A_0x + b_0)' & c_0'x + d_0 + t \end{bmatrix}$$

$$F_i(x) = \begin{bmatrix} I & A_ix + b_i \\ (A_ix + b_i)' & c_i'x + d_i \end{bmatrix}$$

2. Linear fractional programming as SDP: the linear fractional program

$$\begin{aligned} \text{minimize} \quad & (c'x)^2/d'x \\ \text{subj. to} \quad & d'x > 0 \end{aligned}$$

can be expressed by SDP. By Schur's lemma, $(c'x)^2/d'x < t$ is

equivalent to

$$\begin{bmatrix} t & c'x \\ c'x & d'x \end{bmatrix}$$

Therefore, the problem is equivalent to

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subj. to} \quad & \begin{bmatrix} t & c'x \\ c'x & d'x \end{bmatrix} > 0 \end{aligned}$$

3. Riccati inequality: $PA + A'P + Q - (PB + S)'R^{-1}(PB + S) > 0$ with positive definite R; i.e., $R > 0$, is equivalent to

$$\begin{bmatrix} PA + A'P + Q & PB + S \\ B'P + S' & R \end{bmatrix} > 0$$

Applications of SDP

Linear matrix inequality arises in many problems:

- control and system theorem: stability analysis, H_∞, H_2 synthesis, and so on.
- (FIR) filter design.
- civil/mechaincal engineering: structural dynamics optimization
- geometrical problems involving quadratic forms
- combinatorial and convex optimization (Max-Cut problem)
- and many others ...

Differential Inclusion

- A differential inclusion (DI) is described by

$$\dot{x} \in F(x(t), t), \quad x(0) = x_0, \tag{2}$$

  where F is a set-valued function on $\mathbf{R}^n \times \mathbf{R}_+$. Any $x : \mathbf{R}_+ \to \mathbf{R}^n$ that satisfies (2) is called a solution or trajectory of the DI.
- A *linear* differential inclusion (LDI) is described by

$$\dot{x} \in \Omega x, \quad x(0) = x_0, \tag{3}$$

  where Ω is a subset of $\mathbf{R}^{n \times n}$. We can interpret (3) as describing a family of linear time-varying systems. Every trajectory of the LDI satisfies

$$\dot{x} = A(t)x, \quad x(0) = x_0,$$

for some $A : \mathbf{R}_+ \to \Omega$.

We will consider the following type of LDIs in this lecture:

- Linear time invariant systems
- Polytopic linear different inclusion systems
- Norm bounded linear different inclusion systems
- Diagonal norm bounded linear different inclusion systems

Analysis of LDIs: State Properties

Consider the system: $\dot{x} = A(t)x, \quad A(t) \in \Omega$

- LTI systems: in this case, Ω is a singleton $\Omega = \{A\}$.
- Polytopic LDI systems: in this case, Ω is the convex hull

$$\Omega = \mathbf{CO}\{A_1, \cdots, A_n\}$$

- Norm-bounded LDI systems: in this case, we consider systems of the form

$$\begin{aligned} \dot{x} &= Ax + Bp \\ q &= Cx + Dp, \quad p = \Delta(t)q \end{aligned}$$

  where $\|\Delta(t)\| \le 1$ for all $t \ge 0$.

- Diagonal norm-bounded LDI systems: we consider systems of the form

$$\dot{x} = Ax + Bp$$
$$q = Cx + Dp,$$
$$p_i = \delta_i(t)q_i, \quad 1, \cdots, n$$

where $\|\delta_i(t)\| \leq 1$ for all $t \geq 0$.

- Quadratic Stability:
The LDI is stabile if all trajectories of the LDI converge to 0 as $t \to \infty$. A sufficient condition for this is the existence of a quadratic function $V = x'Px$, $P > 0$, that decreases along every nonzero trajectory of the LDI. If such a P exists, we say the LDI is *quadratically stable*, and the corresponding V is called the quadratic Lyapunov function. Since

$$\frac{d}{dt}V(x) = x'(PA(t) + A(t)'P)x,$$

a necessary and sufficient condition for quadratic stability is

$$P > 0, \quad PA + A'P \leq 0, \quad \forall A \in \Omega$$

onditions for quadratic stability of LDIs.

1. Linear time-invariant systems: for a linear time-invariant system, asymptotic stability is equivalent to quadratic stability and the **if and only if** condition for that is: exists $P = P'$ such that

$$P > 0, \quad PA + A'P \leq 0$$

2. Polytopic LDIs: a **if and only if** condition is

$$P > 0, \quad PA_i + A_i'P \leq 0, \quad \forall i = 1, \cdots, n$$

where A_i are the vertices that are used to defined Ω.

Proof. Any $A \in \Omega$ can be expressed as $\sum_{i=1}^{n} \lambda_i A_i$. Hence

$$PA + A'P = \sum_{i=1}^{n} \lambda_i (PA_i + A_i'P)$$

Then the condition is evident. □

. Norm-bounded LDIs: the system is described as

$$\dot{x} = Ax + Bp, \quad q = Cx + Dp, \quad p = \Delta(t)q, \quad \|\Delta(t)\| \leq 1 \qquad (4)$$

A **if and only if** condition for quadratic stability is

$$P > 0, \quad \begin{bmatrix} PA + A'P + C'C & PB + C'D \\ (PB + C'D)' & -(I - D'D) \end{bmatrix} \leq 0$$

4. Diagonal norm-bounded LDIs: the system is similar to (4), except that $\Delta(t)$ is diagonal. A **sufficient** condition for quadratic stability is

$$P > 0, \quad \begin{bmatrix} PA + A'P + C'\Lambda C & PB + C'\Lambda D \\ (PB + C'\Lambda D)' & (D'\Lambda D - \Lambda) \end{bmatrix} \leq 0$$

where $\Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_{n_p})$, $\lambda_i \geq 0$.

- Invariant Ellipsoid:
Quadratic stability can also be interpreted in terms of invariant ellipsoids. For $Q > 0$, let $\mathcal{E}$ denote the ellipsoid centered at the origin

$$\mathcal{E} = \{x \in \mathbf{R}^n \mid x'Q^{-1}x \leq 1\}$$

The ellipsoid is said to be invariant for the LDI if for every trajectory x of the LDI, $x(t_0) \in \mathcal{E}$ implies that $x(t) \in \mathcal{E}$ for all $t \geq t_0$.

Proposition 2. *$\mathcal{E} = \{x \in \mathbf{R}^n \mid x'Q^{-1}x \leq 1\}$ is invariant for LDI $\dot{x} = A(t)x$, $A(t) \in \Omega$ if and only of*

$$QA' + AQ \leq 0, \quad \forall\ A \in \Omega$$

Conditions for checking whether $\mathcal{E}$ is an invariant ellipsoid for a LDI

1. Linear time-invariant systems: $\mathcal{E}$ is an invariant ellipsoid if and only if $Q > 0$, $QA' + AQ \leq 0$.
2. Polytopic LDIs: $\mathcal{E}$ is an invariant ellipsoid if and only if $Q > 0$, $QA_i' + A_iQ \leq 0$, $i = 1, \cdots, n$, where A_i are the vertices of the polytope Ω.
3. Norm-bounded LDIs: $\mathcal{E}$ is an invariant ellipsoid if and only if there exist $\mu \geq 0$ such that

$$Q > 0, \quad \begin{bmatrix} AQ + QA' + \mu BB' & \mu BD' + QC' \\ (\mu BD' + QC')' & -\mu(I - DD') \end{bmatrix} \leq 0$$

Analysis of LDIs: Input/State Properties

Consider the system

$$\dot{x} = A(t)x + B(t)w \qquad \begin{bmatrix} A(t) & B(t) \end{bmatrix} \in \Omega \tag{5}$$

- LTI systems: in this case, Ω is a singleton; i.e.,

$$\Omega = \left\{ \begin{bmatrix} A & B \end{bmatrix} \right\}.$$

- Polytopic LDI systems: in this case, Ω is the convex hull

$$\Omega = \mathbf{CO}\{\begin{bmatrix} A_1 & B_1 \end{bmatrix}, \cdots, \begin{bmatrix} A_n & B_n \end{bmatrix}\}$$

- Norm-bounded LDI systems: in this case, we consider systems of

the form

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2p \\ q &= Cx, \quad p = \Delta(t)q \end{aligned}$$

where $\|\Delta(t)\| \leq 1$ for all $t \geq 0$.

- Diagonal norm-bounded LDI systems: we consider systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2p \\ q &= Cx, \\ p_i &= \delta_i(t)q_i, \quad 1, \cdots, n \end{aligned}$$

where $\|\delta_i(t)\| \leq 1$ for all $t \geq 0$.

- Reachable sets with unit-energy inputs: let $\mathcal{R}_{ue}$ be the set

$$\mathcal{R}_{ue} := \left\{ x(T) \mid x, w \text{ satisfies (5)},\ x(0) = 0,\ \int_0^T w'w dt \leq 1,\ T \geq 0 \right\}$$

Proposition 3. *Let $\mathcal{E} = \{x \mid x'Px \leq 1\}$, where $P = P' > 0$. Consider the function $V = x'Px$. If*

$$\frac{dV(x)}{dt} \leq w'w, \quad \textit{for all } x,\ w \textit{ satisfying (5)} \tag{6}$$

then the ellipsoid $\mathcal{E}$ contains the reachable set $\mathcal{R}_{ue}$.

. LTI systems: an **equivalent** LMI to condition (6)

$$P > 0, \quad \begin{bmatrix} PA + A'P & PB \\ B'P & -I \end{bmatrix} \leq 0$$

2. Polytopic LDI systems: condition (6) holds **if and only if**

$$P > 0, \quad \begin{bmatrix} PA_i + A_i'P & PB_i \\ B_i'P & -I \end{bmatrix} \leq 0,\ i = 1, \cdots, n$$

3. Norm bounded LDIs: condition (6) **is equivalent to** the existence of $P = P'$ and λ such that

$$P > 0, \quad \lambda \geq 0, \quad \begin{bmatrix} PA + A'P + \lambda C'C & PB_1 & PB_2 \\ B_1'P & -I & 0 \\ B_2'P & 0 & -\lambda I \end{bmatrix} \leq 0.$$

4. Diagonal Norm-bounded LDIs: condition (6) **if** there exist of $P = P'$ and $\Lambda = \textbf{diag}(\lambda_1, \cdots, \lambda_n)$ such that

$$P > 0, \quad \Lambda \geq 0, \quad \begin{bmatrix} PA + A'P + C'\Lambda C & PB_1 & PB_2 \\ B_1'P & -I & 0 \\ B_2'P & 0 & -\Lambda \end{bmatrix} \leq 0.$$

- Reachable sets with unit-peak inputs: let $\mathcal{R}_{up}$ be the set

$$\mathcal{R}_{up} := \{x(T) \mid x, w \text{ satisfies (5)},\ x(0) = 0,\ \|w(t)\| \leq 1,\ \forall\ t \geq 0,\ T \geq 0\}$$

Proposition 4. *Let $\mathcal{E} = \{x \mid x'Px \leq 1\}$, where $P = P' > 0$. Consider the function $V = x'Px$. If*

$$\frac{dV(x)}{dt} \leq 0, \quad \textit{for all } x,\ w \textit{ satisfying (5)}, \quad w'w \leq 1,\ \textit{and } V(x) \geq 1, \tag{7}$$

then the ellipsoid $\mathcal{E}$ contains the reachable set $\mathcal{R}_{up}$.

1. LTI systems: condition (7) is satisfies if there exist $P = P' > 0$, $\alpha \geq 0$ such that

$$\begin{bmatrix} PA + A'P + \alpha P & PB \\ B'P & -\alpha I \end{bmatrix} \leq 0$$

2. Polytopic LDI systems: condition (7) is satisfies if there exist $P = P' > 0$, $\alpha \geq 0$ such that

$$\begin{bmatrix} PA_i + A_i'P + \alpha P & PB_i \\ B_i'P & -\alpha I \end{bmatrix} \leq 0, \quad i = 1, \cdots, n$$

3. Norm bounded LDI systems: condition (7) is satisfies if there

exist $P = P' > 0$, $\alpha \geq 0$, $\lambda \geq 0$ such that

$$\begin{bmatrix} PA + A'P + \alpha P + \lambda C'C & PB_1 & PB_2 \\ B_1'P & -\alpha I & 0 \\ B_2'P & 0 & -\lambda I \end{bmatrix} \leq 0,$$

4. Diagonal Norm-bounded LDIs: condition (7) is satisfies if there exist $P = P' > 0$, $\alpha \geq 0$, $\Lambda \geq 0$ such that

$$\begin{bmatrix} PA + A'P + \alpha P + C'\Lambda C & PB_1 & PB_2 \\ B_1'P & -\alpha I & 0 \\ B_2'P & 0 & -\Lambda \end{bmatrix} \leq 0,$$

where $\Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_n)$.

State/Output Properties

We now consider the state-to-output properties for the LDI

$$\begin{aligned} \dot{x}(t) &= A(t)x \\ z(t) &= C(t)x' \end{aligned} \quad \begin{bmatrix} A(t) \\ C(t) \end{bmatrix} \in \Omega. \tag{8}$$

- LTI systems: $\dot{x} = Ax$, $z = Cx$.
- Polytopic LDI systems: Ω has the form

$$\Omega = \mathbf{CO}\left\{ \begin{bmatrix} A_1 \\ C_1 \end{bmatrix}, \cdots, \begin{bmatrix} A_n \\ C_n \end{bmatrix} \right\}$$

- Norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + Bp, \quad z = C_1 x \\ q &= C_2 x, \\ p &= \Delta(t) q \end{aligned}$$

where $\|\Delta(t)\| \leq 1$ for all $t \geq 0$.

- Diagonal norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + Bp, \quad z = C_1 x \\ q &= C_2 x, \\ p_i &= \delta_i(t) q_i \end{aligned}$$

where $\|\delta_i(t)\| \leq 1$ for all $t \geq 0$, $i = 1, \cdots, n$.

- Bounds on output energy: we seek the maximum output energy given a certain initial state,

$$\max \left\{ \int_0^\infty z'z dt \mid \dot{x} = A(t)x, \quad z = C(t)x \right\}$$

where $x(0) = x_0$ is given, and the maximum is taken over $(A(t), C(t))$ such that (8) holds.

Proposition 5. *Let $V = x'Px$. If*

$$\frac{dV(x)}{dt} \leq -z'z, \quad \text{for all } x \text{ and } z \text{ satisfying (8)}, \tag{9}$$

then the energy of z is bounded by $x_0'Px_0$.

1. LTI systems: condition (9) **is equivalent to** the existence of $P = P'$ such that $PA + A'P + C'C \leq 0$.
2. Polytopic LDI systems: condition (9) **is equivalent to** the existence of $P = P'$ such that $PA_i + A_i'P + C_i'C_i \leq 0$, $i = 1, \cdots, n$.
3. Norm-bounded LDI systems: condition (9) **is equivalent to** the existence of $P = P'$ and $\lambda \geq 0$ such that

$$\begin{bmatrix} PA + A'P + C_1'C_1 + \lambda C_2'C_2 & PB \\ B'P & -\lambda I \end{bmatrix} \leq 0$$

4. Diagonal norm-bounded LDI systems: condition (9) hold **if** there

exist of $P = P'$ and $\Lambda \geq 0$ such that

$$\begin{bmatrix} PA + A'P + C_1'C_1 + C_2'\Lambda C_2 & PB \\ B'P & -\Lambda \end{bmatrix} \leq 0$$

where $\Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_n)$.

- Bounds on output amplitude: it is possible to derive bounds on $\|z(t)\|$ using invariant ellipsoids. Assume that the initial state x_0 is known. Suppose $\mathcal{E} := \{x \mid x'Px \leq 1\}$ is an invariant ellipsoid containing x_0 for the LDI (8). Then

$$z'(t)z(t) \leq \max_{x \in \mathcal{E}} x'C(t)'C(t)x$$

for all $t \geq 0$. We can express $\max_{x \in \mathcal{E}} x'C(t)'C(t)x$ as the square root of the minimum γ subject to

$$\begin{bmatrix} P & C' \\ C & \gamma I \end{bmatrix} \geq 0 \tag{10}$$

. LTI systems:

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subj. to } P > 0, \quad x(0)'Px(0) \leq 1, \\ &PA + A'P \leq 0, \quad \begin{bmatrix} P & C' \\ C & \gamma I \end{bmatrix} \geq 0. \end{aligned}$$

2. Polytopic LDI systems:

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subj. to } P > 0, \quad x(0)'Px(0) \leq 1, \\ &PA_i + A_i'P \leq 0, \quad \begin{bmatrix} P & C_i' \\ C_i & \gamma I \end{bmatrix} \geq 0, \quad i = 1, \cdots, n \end{aligned}$$

. Norm-bounded LDI systems:

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subj. to } P > 0, \quad \lambda \geq 0, \quad x(0)'Px(0) \leq 1, \\ &\begin{bmatrix} PA + A'P + \lambda C_2'C_2 & PB \\ B'P & -\lambda I \end{bmatrix} \leq 0, \quad \begin{bmatrix} P & C_1' \\ C_1 & \gamma I \end{bmatrix} \geq 0 \end{aligned}$$

4. Diagonal norm-bounded LDI systems:

$$\begin{aligned} &\text{minimize } \gamma \\ &\text{subj. to } P > 0, \quad \Lambda \geq 0, \quad x(0)'Px(0) \leq 1, \\ &\begin{bmatrix} PA + A'P + C_2'\Lambda C_2 & PB \\ B'P & -\Lambda \end{bmatrix} \leq 0, \quad \begin{bmatrix} P & C_1' \\ C_1 & \gamma I \end{bmatrix} \geq 0 \end{aligned}$$

Input/Output Properties

We now consider LDI of the following form:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)w \\ z &= C(t)x + D(t)w \end{aligned}, \quad \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix} \in \Omega \tag{11}$$

1. LTI systems: in this case, Ω has the form

$$\Omega = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\}$$

2. Polytopic LDI systems: in this case, Ω has the form

$$\Omega = \mathbf{Co} \left\{ \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \cdots, \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \right\}$$

3. Norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 p \\ z &= C_1 x + D_1 w, \quad q = C_2 x \\ p &= \Delta(t) q \end{aligned}$$

where $\|\Delta(t)\| \leq 1$, for all t.

4. Diagonal norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 p \\ z &= C_1 x + D_1 w, \quad q = C_2 x \\ p_i &= \delta_i(t) q_i \end{aligned}$$

where $|\delta_i(t)| \leq 1$, for all t.

- $\mathcal{L}_2$ gain: the $\mathcal{L}_2$ of the LDI system is defined as the quantity $\sup_{\|w\|_{L_2}\neq 0} \|z\|_{L_2}/\|w\|_{L_2}$, where the $\mathcal{L}_2$ norm of a signal w is

$$\|w\|_{L_2} = \int_0^\infty w'wdt,$$

and the supremum if taken over all non-zero trajectories of the LDI, starting from $x(0) = 0$.

Proposition 6. *Suppose there exists a quadratic function $V(x) = x'Px$, where $P = P' > 0$, and a constant $\gamma \geq 0$ such that for all t,*

$$\frac{dV(x)}{dt} + z'z - \gamma^2 w'w \leq 0, \quad \forall\ x \text{ and } w \text{ satisfying (11)} \qquad (12)$$

Then the $\mathcal{L}_2$ gain of the LDI system is less than γ.

1. LTI systems: condition (12) **is equivalent to**

$$\begin{bmatrix} PA + A'P + C'C & PB + C'D \\ B'P + D'C & -\gamma^2 I + D'D \end{bmatrix} \leq 0$$

Remark 1. This is the LMI condition for the $\mathcal{H}_\infty$ norm of $C(sI - A)^{-1}B + D$ to be less than or equal to γ. Also, we know that this problem has analytical solutions.

2. Polytopic LDI systems: condition (12) **is equivalent to**

$$\begin{bmatrix} PA_i + A_i'P + C_i'C_i & PB_i + C_i'D_i \\ B_i'P + D_i'C_i & -\gamma^2 I + D_i'D_i \end{bmatrix} \leq 0, \quad i = 1, \cdots, n$$

3. Norm-bounded LDI systems: condition (12) **is equivalent to**

$$\begin{bmatrix} PA_i + A_i'P + C_1'C_1 + \lambda C_2'C_2 & PB_1 + C_1'D_1 & PB_2 \\ B_1'P + D_1'C_1 & -\gamma^2 I + D_1'D_1 & 0 \\ B_2'P & 0 & -\lambda I \end{bmatrix} \leq 0.$$

4. Diagonal norm-bounded LDI systems: condition (12) holds **if**

$$\begin{bmatrix} PA_i + A_i'P + C_1'C_1 + C_2'\Lambda C_2 & PB_1 + C_1'D_1 & PB_2 \\ B_1'P + D_1'C_1 & -\gamma^2 I + D_1'D_1 & 0 \\ B_2'P & 0 & -\Lambda \end{bmatrix} \leq 0.$$

where $\Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_n)$, and $\Lambda \geq 0$.

- Dissipativity: the LDI (11) is said to be passive if every solution with $x(0) = 0$ satisfies

$$\int_0^T w'zdt \geq 0$$

for all $T \geq 0$. It is said to have dissipation γ if

$$\int_0^T w'zdt \geq \gamma \int_0^T w'wdt$$

for all $T \geq 0$. Thus the passivity corresponds to non-negative dissipation. The largest dissipation of the system is called its *dissipativity.*

Proposition 7. *Suppose there is a function $V(x) = x'Px$, $P = P' > 0$ such that*

$$\frac{dV(x)}{dt} - 2w'z + 2\gamma w'w \leq 0, \quad \textit{for all } x,\ w \textit{ satisfying (11)} \qquad (13)$$

then the dissipativity of the LDI (11) is at least γ.

1. LTI systems: condition (13) **is equivalent to**

$$\begin{bmatrix} PA + A'P & PB - C' \\ B'P - C & 2\gamma I - (D' + D) \end{bmatrix} \leq 0$$

2. Polytopic LDI systems: condition (12) **is equivalent to**

$$\begin{bmatrix} PA_i + A_i'P & PB_i - C_i' \\ B_i'P - C_i & 2\gamma I - (D_i' + D_i) \end{bmatrix} \leq 0, \quad i = 1, \cdots, n$$

3. Norm-bounded LDI systems: condition (12) **is equivalent to**

$$\begin{bmatrix} PA_i + A_i'P + \lambda C_2'C_2 & PB_1 - C_1' & PB_2 \\ B_1'P - C_1 & 2\gamma I - (D_1 + D_1') & 0 \\ B_2'P & 0 & -\lambda I \end{bmatrix} \leq 0.$$

4. Diagonal norm-bounded LDI systems: condition (12) holds **if**

$$\begin{bmatrix} PA_i + A_i'P + C_2'\Lambda C_2 & PB_1 - C_1' & PB_2 \\ B_1'P - C_1 & 2\gamma I - (D_1 + D_1') & 0 \\ B_2'P & 0 & -\Lambda \end{bmatrix} \leq 0.$$

where $\Lambda = \mathbf{diag}(\lambda_1, \cdots, \lambda_n)$, and $\Lambda \geq 0$.