Linear Matrix Inequalities in Control System Synthesis

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Synthesis of LDIs: State Properties

Consider the system: $\dot{x} = A(t)x + B_u(t)u, \quad \begin{bmatrix} A(t) & B(t) \end{bmatrix} \in \Omega$

- LTI systems: in this case, we have $\dot{x} = Ax + B_u u$.
- Polytopic LDI systems: in this case, Ω is the convex hull

$$\Omega = \mathbf{CO}\{\begin{bmatrix} A_1 & B_{u,1} \end{bmatrix}, \cdots, \begin{bmatrix} A_n & B_{u,n} \end{bmatrix}\}$$

- Norm-bounded LDI systems: in this case, we consider systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_p p \\ q &= Cx + D_u u + D_p p, \quad p = \Delta(t) q \end{aligned}$$

where $\|\Delta(t)\| \leq 1$ for all $t \geq 0$.

- Diagonal norm-bounded LDI systems: we consider systems of the form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_p p \\ q &= Cx + D_u u + D_p p, \\ p_i &= \delta_i(t) q_i, \quad 1, \cdots, n \end{aligned}$$

where $\|\delta_i(t)\| \leq 1$ for all $t \geq 0$.

- Quadratic Stability: recall that the necessary and sufficient condition for quadratic stability of a LDI is

$$P > 0, \quad PA(t) + A(t)'P \leq 0, \quad \forall A(t) \in \Omega$$

Conditions for quadratic stability of LDIs.

1. Linear time-invariant systems: quadratic stabile **if and only if** there exist $Q = Q' > 0$ and Y such that

$$AQ + QA' + B_u Y + Y' B_u' \leq 0$$

Proof. We have to find $P = P' > 0$ and K such that

$$P(A + B_u K) + (A + B_u K)'P \leq 0$$
$$\Updownarrow$$
$$PA + +A'P + PB_u K + K'B_u'P \leq 0$$
$$\Updownarrow$$
$$AP^{-1} + +P^{-1}A' + B_u KP^{-1} + P^{-1}K'B_u' \leq 0$$

Let $Q = P^{-1}$ and $Y = KP^{-1}$. $\square$

2. Polytopic LDIs: quadratic stabile **if and only if** there exist $Q = Q' > 0$ and Y such that

$$Q = Q' > 0, \quad A_i Q + QA_i' + +B_{u,i}Y + Y'B_{u,i}' \leq 0, \quad \forall i = 1, \cdots, n$$

3. Norm-bounded LDIs: quadratic stabile if there exist $Q = Q' > 0$,

Y, and $\mu > 0$ such that

$$\begin{bmatrix} AQ + QA' + B_u Y + Y'B_u' + \mu B_p B_p' & \mu B_p D_p' + QC' + Y'D_u' \\ \mu D_p B_p' + CQ + D_u Y & \mu(D_p D_p' - I) \end{bmatrix} \leq 0$$

4. Diagonal norm-bounded LDIs: quadratic stabile if there exist $Q = Q' > 0$, Y, and $\Pi > 0$ such that

$$\begin{bmatrix} AQ + QA' + B_u Y + Y'B_u' + B_p \Pi B_p' & B_p \Pi D_p' + QC' + Y'D_u' \\ D_p \Pi B_p' + CQ + D_u Y & D_p \Pi D_p' - \Pi \end{bmatrix} \leq 0$$

where $\Pi = \mathbf{diag}(\mu_1, \cdots, \mu_n)$.

Synthesis of LDIs: Input/State Properties

Consider the system

$$\dot{x} = A(t)x + B_u(t)u + B_w(t)w, \quad \begin{bmatrix} A(t) & B_u(t) & B_w(t) \end{bmatrix} \in \Omega \qquad (1)$$

- LTI systems: $\dot{x} = Ax + B_u u + B_w w$, $x(0) = 0$.
- Polytopic LDI systems: in this case, Ω is the convex hull

$$\Omega = \mathbf{CO}\{\begin{bmatrix} A_1 & B_{u,1} & B_{w,1} \end{bmatrix}, \cdots, \begin{bmatrix} A_n & B_{u,n} & B_{w,n} \end{bmatrix}\}$$

- Norm-bounded LDI systems: in this case, we consider systems of the form

$$\dot{x} = Ax + B_u u + B_w w + B_p p$$
$$q = Cx + D_p p, \quad p = \Delta(t) q$$

where $\|\Delta(t)\| \leq 1$ for all $t \geq 0$.

- Diagonal norm-bounded LDI systems: we consider systems of the form

$$\dot{x} = Ax + B_u u + B_w w + B_p p$$
$$q = Cx + D_p p,$$
$$p_i = \delta_i(t) q_i, \quad 1, \cdots, n$$

where $\|\delta_i(t)\| \leq 1$ for all $t \geq 0$.

- Reachable sets with unit-energy inputs: let $\mathcal{R}_{ue}$ be the set

$$\mathcal{R}_{ue} := \left\{ x(T) \mid x, u, w \text{ satisfies (1)},\ u = Kx,\ x(0) = 0, \int_0^T w'w dt \le 1,\ T \ge 0 \right\}$$

We want to find a feedback control $u = Kx$ and $\mathcal{E} = \{x \mid x'Px \le 1\}$, such that $\mathcal{R}_{ue} \subset \mathcal{E}$ is guaranteed.

1. LTI systems: if there exist $Q = Q' > 0$ and Y such that

$$\begin{bmatrix} AQ + QA' + B_uY + Y'B_u' & B_w \\ B_w' & -I \end{bmatrix} \le 0$$

then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x'Q^{-1}x \le 1\}$.

2. Polytopic LDI systems: if there exist $Q = Q' > 0$ and Y such that

$$\begin{bmatrix} A_iQ + QA_i' + B_{u,i}Y + Y'B_{u,i}' & B_{w,i} \\ B_{w,i}' & -I \end{bmatrix} \le 0, \quad i = 1, \cdots, n$$

then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x'Q^{-1}x \le 1\}$.

3. Norm bounded LDIs: if there exist $Q = Q' > 0$, Y, $\mu > 0$ such

that

$$\begin{bmatrix} A_iQ + QA_i' + B_uY + Y'B_u' + \mu B_pB_p' & QC' + \mu B_pD_p' & B_w \\ CQ + \mu D_pB_p' & D_pD_p' - \mu I & 0 \\ B_w' & 0 & -I \end{bmatrix} \le 0$$

then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x'Q^{-1}x \le 1\}$.

4. Diagonal Norm-bounded LDIs: suppose that there exist $Q = Q' > 0$, Y, and $\Pi > 0$ such that

$$\begin{bmatrix} A_iQ + QA_i' + B_uY + Y'B_u' + B_p\Pi B_p' & QC' + B_p\Pi D_p' & B_w \\ CQ + D_p\Pi B_p' & D_p\Pi D_p' - \Pi & 0 \\ B_w' & 0 & -I \end{bmatrix} \le 0$$

where $\Pi = \mathbf{diag}(\mu_1, \cdots, \mu_n)$. Then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x'Q^{-1}x \le 1\}$.

- Reachable sets with unit-peak inputs: let $\mathcal{R}_{up}$ be the set

$$\mathcal{R}_{up} := \{ x(T) \mid x, u, w \text{ satisfies (1)},\ u = Kx,\ x(0) = 0, \|w(t)\| \le 1,\ \forall\, t \ge 0, \quad T \ge 0 \}$$

We want to find a feedback control $u = Kx$ and $\mathcal{E} = \{x \mid x'Px \le 1\}$, such that $\mathcal{R}_{up} \subset \mathcal{E}$ is guaranteed.

1. LTI systems: if there exist $Q = Q' > 0$, Y, and $\alpha > 0$ such that

$$\begin{bmatrix} AQ + QA' + B_u Y + Y' B_u' + \alpha Q & B_w \\ B_w' & -\alpha I \end{bmatrix} \leq 0$$

then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x' Q^{-1} x \leq 1\}$.

2. Polytopic LDI systems: if there exist $Q = Q' > 0$, Y, and $\alpha > 0$ such that

$$\begin{bmatrix} A_i Q + QA_i' + B_{u,i} Y + Y' B_{u,i}' + \alpha Q & B_{w,i} \\ B_{w,i}' & -\alpha I \end{bmatrix} \leq 0, \quad i = 1, \cdots, n$$

then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x' Q^{-1} x \leq 1\}$.

3. Norm bounded LDIs: if there exist $Q = Q' > 0$, Y, $\mu > 0$ and

$\alpha > 0$ such that

$$\begin{bmatrix} A_i Q + QA_i' + B_u Y + Y' B_u' + \alpha Q + \mu B_p B_p' & QC' + \mu B_p D_p' & B_w \\ CQ + \mu D_p B_p' & D_p D_p' - \mu I & 0 \\ B_w' & 0 & -\alpha I \end{bmatrix} \leq 0$$

then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x' Q^{-1} x \leq 1\}$.

4. Diagonal Norm-bounded LDIs: suppose that there exist $Q = Q' > 0$, Y, $\alpha > 0$ and $\Pi > 0$ such that

$$\begin{bmatrix} A_i Q + QA_i' + B_u Y + Y' B_u' + B_p \Pi B_p' + \alpha Q & QC' + B_p \Pi D_p' & B_w \\ CQ + D_p \Pi B_p' & D_p \Pi D_p' - \Pi & 0 \\ B_w' & 0 & -\alpha I \end{bmatrix} \leq 0$$

where $\Pi = \mathbf{diag}(\mu_1, \cdots, \mu_n)$. Then $\mathcal{R}_{ue} \subset \mathcal{E} := \{x \mid x' Q^{-1} x \leq 1\}$.

State/Output Properties

We now consider the state-to-output properties for the LDI

$$\begin{aligned} \dot{x}(t) &= A(t)x + B_u(t)u, \quad x(0) = x_0 \\ z(t) &= C(t)x + D_u(t)u \end{aligned} \qquad \begin{bmatrix} A(t) & B_u(t) \\ C(t) & D_u(t) \end{bmatrix} \in \Omega. \qquad (2)$$

- LTI systems: consider the systems:

$$\begin{aligned} \dot{x} &= Ax + B_u u, \quad x(0) = x_0 \\ z &= Cx + D_u u. \end{aligned}$$

- Polytopic LDI systems: Ω has the form

$$\Omega = \mathbf{CO}\left\{ \begin{bmatrix} A_1 & B_{u,1} \\ C_1 & D_{u,1} \end{bmatrix}, \cdots, \begin{bmatrix} A_n & B_{u,n} \\ C_n & D_{u,n} \end{bmatrix} \right\}$$

- Norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_p p, \quad x(0) = x_0 \\ z &= C_z x + D_u u \\ q &= C_q x, \quad p = \Delta(t) q, \quad \|\Delta(t)\| \leq 1, \quad \forall\, t. \end{aligned}$$

- Diagonal norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_p p, \quad x(0) = x_0 \\ z &= C_z x + D_u u \\ q &= C_q x, \quad p_i = \delta_i(t) q_i, \quad \|\delta_i(t)\| \leq 1, \ \forall\, t, \ i = 1, \cdots, n \end{aligned}$$

- Bounds on output energy: given the initial state x_0 we seek a state feedback control $u = Kx$ such that the maximum output energy is bounded by some specific number,

Recall: Let $u = Kx$, and $V = x'Px$, where $P = P' > 0$. If

$$\frac{dV(x)}{dt} \leq -z'z,$$

for all (x, z) satisfies the closed loop dynamics. Then the energy of z is bounded by $x_0'Px_0$.

. LTI systems: $dV/dt + z'z \leq 0$ along the closed-loop dynamics gives the following LMI condition:

$$P(A + B_uK) + (A + B_uK)'P + (C + D_uK)'(C + D_uK) \leq 0$$

$$\Updownarrow$$

$$\begin{bmatrix} PA + A'P + PB_uK + K'B_u'P & C' + K'D_u' \\ C + D_uK & -I \end{bmatrix} \leq 0$$

$$\Updownarrow$$

$$\begin{bmatrix} AQ + QA' + B_uY + Y'B_u' & QC' + Y'D_u' \\ CQ + D_uY & -I \end{bmatrix} \leq 0$$

where $Q = P^{-1}$ and $Y = KP^{-1}$. The energy of z is bounded by $x_0'Px_0 \equiv x_0'Q^{-1}x_0$. Hence, given γ, finding a state feedback control such that the output energy is bounded by γ can be post

as LMI problem: find $Q = Q' > 0$ and Y such that

$$\begin{bmatrix} AQ + QA' + B_uY + Y'B_u' & QC' + Y'D_u' \\ CQ + D_uY & -I \end{bmatrix} \leq 0, \quad \begin{bmatrix} \gamma & x_0' \\ x_0 & Q \end{bmatrix} \geq 0,$$

2. Polytopic LDI systems: the output energy is bounded by γ if there exists $Q = Q' > 0$, and Y such that

$$\begin{bmatrix} \gamma & x_0' \\ x_0 & Q \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} A_iQ + QA_i' + B_{u,i}Y + Y'B_{u,i}' & QC_i' + Y'D_{u,i}' \\ C_iQ + D_{u,i}Y & -I \end{bmatrix} \leq 0, \quad i = 1, \cdots, n$$

3. Norm-bounded LDI systems: the output energy is bounded by γ

if there exists $Q = Q' > 0$, Y and $\mu > 0$ such that

$$\begin{bmatrix} \gamma & x_0' \\ x_0 & Q \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} AQ + QA' + B_uY + Y'B_u' + \mu B_pB_p' & QC_z' + Y'D_u' & QC_q' \\ C_zQ + D_uY & -I & 0 \\ C_qQ & 0 & -\mu I \end{bmatrix} \leq 0$$

4. Diagonal norm-bounded LDI systems: the output energy is

bounded by γ if there exists $Q = Q' > 0$, Y, and $\Pi > 0$ such that

$$\begin{bmatrix} \gamma & x_0' \\ x_0 & Q \end{bmatrix} \geq 0,$$

$$\begin{bmatrix} AQ + QA' + B_u Y + Y' B_u' + B_p \Pi B_p' & QC_z' + Y' D_u' & QC_q' \\ C_z Q + D_u Y & -I & 0 \\ C_q Q & 0 & -\Pi \end{bmatrix} \leq 0$$

where $\Pi = \mathbf{diag}(\mu_1, \cdots, \mu_n)$.

Input/Output Properties

We now consider LDI of the following form:

$$\begin{aligned} \dot{x} &= A(t)x + B_u(t)u + B_w(t)w \\ z &= C(t)x + D_u(t)u + D_w(t)w \end{aligned}, \quad \begin{bmatrix} A(t) & B_u(t) & B_w(t) \\ C(t) & D_u(t) & D_w(t) \end{bmatrix} \in \Omega \quad (3)$$

1. LTI systems: in this case, Ω has the form

$$\Omega = \left\{ \begin{bmatrix} A & B_u & B_w \\ C & D_u & D_w \end{bmatrix} \right\}$$

2. Polytopic LDI systems: in this case, Ω has the form

$$\Omega = \mathbf{Co} \left\{ \begin{bmatrix} A_1 & B_{u,1} & B_{w,1} \\ C_1 & D_{u,1} & D_{w,1} \end{bmatrix}, \cdots, \begin{bmatrix} A_n & B_{u,n} & B_{w.n} \\ C_n & D_{u,n} & D_{w,n} \end{bmatrix} \right\}$$

3. Norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_w w + B_p p, \quad x(0) = 0 \\ z &= C_z x + D_u u + D_w w, \\ q &= C_q x, \quad p = \Delta(t) q \end{aligned}$$

where $\|\Delta(t)\| \leq 1$, for all t.

4. Diagonal norm-bounded LDI systems: the systems have the form

$$\begin{aligned} \dot{x} &= Ax + B_u u + B_w w + B_p p, \quad x(0) = 0 \\ z &= C_z x + D_u u + D_w w, \\ q &= C_q x, \quad p_i = \delta_i(t) q_i \end{aligned}$$

where $|\delta_i(t)| \leq 1$, for all t.

- Bound on the $\mathcal{L}_2$ gain: we wand to find a state feedback control $u = Kx$ such that

$$\int_0^\infty z(t)' z(t) dt \leq \gamma^2 \int_0^\infty w(t)' w(t) dt,$$

for all finite energy input w, where γ is a given number.

Recall: Consider $u = Kx$ and the closed loop dynamics of 3. Suppose there exists a quadratic function $V(x) = x' P x$, where $P = P' > 0$ such that

$$\frac{dV(x)}{dt} + z'z - \gamma^2 w' w \leq 0,$$

for all (z, w) satisfying the closed loop dynamical equation. Then the $\mathcal{L}_2$ gain of the LDI system is less than or equal to γ; i.e.,

$$\int_0^\infty z(t)' z(t) dt \leq \gamma^2 \int_0^\infty w(t)' w(t) dt,$$

for all finite energy input w.

. LTI systems: there exists state feedback gain K such that the $\mathcal{L}_2$ gain of the system is less than γ if there exists $Q = Q' > 0$ and Y such that

$$\begin{bmatrix} AQ + QA' + B_u Y + Y' B_u' + B_w B_w' & QC' + Y' D_u' + B_w D_w' \\ CQ + D_u Y + D_w B_w' & -\gamma^2 I + D_w D_w' \end{bmatrix} \leq 0$$

2. Polytopic LDI systems: there exists state feedback gain K such that the $\mathcal{L}_2$ gain of the system is less than γ if there exists $Q = Q' > 0$ and Y such that

$$\begin{bmatrix} A_i Q + QA_i' + B_{u,i} Y + Y' B_{u,i}' + B_{w,i} B_{w,i}' & \star \\ CQ + D_{u,i} Y + D_{w,i} B_{w,i}' & -\gamma^2 I + D_{w,i} D_{w,i}' \end{bmatrix} \leq 0$$

for $i = 1, \cdots, n$, where $\star := (CQ + D_{u,i} Y + D_{w,i} B_{w,i}')'$.

3. Norm-bounded LDI systems: there exists state feedback gain K

such that the $\mathcal{L}_2$ gain of the system is less than γ if there exists $Q = Q' > 0$, Y, and $\mu > 0$ such that

$$\begin{bmatrix} AQ + QA' + B_u Y + Y' B_u' + B_w B_w' + \mu B_p B_p' & \Theta_1 & QC_q' \\ \Theta_1' & \Theta_2 & 0 \\ C_q Q & 0 & -\mu I \end{bmatrix} \leq 0$$

where $\Theta := QC_z' + Y' D_u' + B_w D_w'$, $\Theta_2 = -\gamma^2 I + D_w D_w'$.