Control Synthesis Using LMIs

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Output Feedback Synthesis

Consider linear time invariant system:

$$\dot{x} = Ax + Bu, \quad y = Cx$$

(A, B) controllable, (A, C) observable.

- Static output feedback problem: find a matrix K such that $u = Ky$ stabilize the system; i.e., $A_{cl} := A + BKC$ be a stable matrix.
- Dynamic output feedback problem: find a controller

$$\dot{x}_K = A_K x_K + B_K y$$
$$u = C_K x_K + D_K y$$

such that the closed loop system is stable.

dynamical feedback case, the closed loop system matrix becomes

$$A_{cl} = \begin{bmatrix} A + BD_K C & BC_K \\ B_K C & A_K \end{bmatrix}$$

Let us define

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix}, \ \mathcal{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$

Then $A_{cl} = \mathcal{A} + \mathcal{B}K\mathcal{C}$, where

$$K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$

is the matrix of controller parameters

Design for Quadratic Stability

Simplest design objective is stability of A_{cl}

Theorem 1 (Lyapunov). *The following are equivalent*

1. *A_{cl} is stable (real parts of eigenvalues of A_{cl} are less than 0)*
2. *There exists $P = P' > 0$ such that $PA_{cl} + A_{cl}'P < 0$*

Design Problem: Find P and K such that

$$P(\mathcal{A} + \mathcal{B}K\mathcal{C}) + (\mathcal{A} + \mathcal{B}K\mathcal{C})'P < 0$$

- Not a convex problem because of $P\mathcal{B}K\mathcal{C}$ terms.
- The "multiplying P^{-1} from both sides" trick does not work here.

Elimination Lemma

Definition 1. An orthogonal complement $N_\perp$ is any matrix of maximal rank such that $N_\perp N = 0$ and $N_\perp N_\perp' > 0$.

Lemma 1. *Let $M \in \boldsymbol{R}^{n\times k}$, $N \in \boldsymbol{R}^{n\times m}$, and $H = H' \in \boldsymbol{R}^{n\times n}$. The following statements are equivalent*

(i) There exists $X \in \boldsymbol{R}^{m\times k}$ such that

$$NXM' + MX'N' + H < 0$$

(ii) The following two conditions hold

$$N_\perp H N_\perp' < 0 \quad \text{or} \quad NN' > 0$$
$$M_\perp H M_\perp' < 0 \quad \text{or} \quad MM' > 0$$

Proof. Obviously, the first statement implies the second one. Let's prove the converse. Suppose the second statement holds but not the first one. This means for all X, $NXM' + MX'N' + H \not< 0$. By duality, this in turn implies that there exists $Z \neq 0$, $Z \geq 0$, such that $M'ZN = 0$ and $\mathbf{tr}HZ \leq 0$.

Let $Z = RR'$. The condition $M'ZN = 0$ is equivalent to

$$M'RR'N = (M'R)(N'R)' = 0$$

This implies that there exists a unitary matrix T and matrix V and U such that

$$M'RT = \begin{bmatrix} 0 & V \end{bmatrix}, \quad N'RT = \begin{bmatrix} U & 0 \end{bmatrix},$$

where the columns of U and V are added up to that of R. In other words, we have $RT = \begin{bmatrix} A & B \end{bmatrix}$ such that $M'A = 0$, $N'B = 0$. From the definition of $M_\perp$ and $N_\perp$, we can write matrix A and B as

$A = M_\perp' P$, $B = N_\perp' Q$. for some matrices P and Q. Hence

$$\begin{aligned} Z = RR' &= (RT)(RT)' = AA' + BB' \\ &= M_\perp' PP' M_\perp + N_\perp' QQ' N_\perp. \end{aligned}$$

Hence,

$$\mathbf{tr}(HZ) = \mathbf{tr}(M_\perp H M_\perp' PP') + \mathbf{tr}(N_\perp H N_\perp' QQ') > 0$$

which contradicts $\mathbf{tr}(HZ) \leq 0$. This concludes the proof. □

applied to our problem:

$$P(\mathcal{A} + \mathcal{B}K\mathcal{C}) + (\mathcal{A} + \mathcal{B}K\mathcal{C})'P < 0$$
$$\Updownarrow$$
$$\underbrace{(P\mathcal{B})}_{N}\underbrace{K}_{X}\underbrace{\mathcal{C}}_{M'} + \mathcal{C}'K'(\mathcal{B}'P) + \underbrace{P\mathcal{A} + \mathcal{A}'P}_{H} < 0$$

By elimination lemma this is equivalent to

$$N_\perp(P\mathcal{A} + \mathcal{A}'P)(N_\perp)' < 0$$
$$M_\perp(P\mathcal{A} + \mathcal{A}'P)(M_\perp)' < 0$$

- $N_\perp = \mathcal{B}_\perp P^{-1} = \begin{bmatrix} B_\perp & 0 \end{bmatrix} P^{-1}$
- $M_\perp = (\mathcal{C}')_\perp = \begin{bmatrix} (C')_\perp & 0 \end{bmatrix}$

KTH

Equivalent stabilization problem:

Find $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ such that

$$\begin{bmatrix} B_\perp & 0 \end{bmatrix} \left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} P^{-1} + P^{-1} \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} B_\perp' \\ 0 \end{bmatrix} < 0$$

$$\begin{bmatrix} (C')_\perp & 0 \end{bmatrix} \left(P \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} A' & 0 \\ 0 & 0 \end{bmatrix} P \right) \begin{bmatrix} ((C')_\perp)' \\ 0 \end{bmatrix} < 0$$

- Still NOT convex (because we have both P and P^{-1} involved).
- We "fix" this by introducing block structure for P and P^{-1}

$$P = \begin{bmatrix} X & X_2 \\ X_2' & X_3 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2' & Y_3 \end{bmatrix}$$

KTH

Let n_G be the dimension of A, and n_K be the dimension of A_K. The next lemma shows that it is enough to find $X, Y \in \mathcal{S}_+^{n_G \times n_G}$.

Lemma 2. *Suppose $X, Y \in \mathcal{S}_+^{n_G \times n_G}$. The following are equivalent*

(i) There exists $X_2, Y_2 \in \boldsymbol{R}^{n_G \times n_K}$ and $X_3, Y_3 \in \mathcal{S}_+^{n_K \times n_K}$ s.t.

$$\begin{bmatrix} X & X_2 \\ X_2' & X_3 \end{bmatrix} > 0 \quad \text{and} \quad \begin{bmatrix} X & X_2 \\ X_2' & X_3 \end{bmatrix}^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2' & Y_3 \end{bmatrix} \tag{1}$$

(ii) We have

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \quad \text{and} \quad \mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K \tag{2}$$

KTH

Stabilization Theorem

Theorem 2. *The following are equivalent*

- *$G(s) = C(sI - A)^{-1}B$ can be stabilized by a n_K-dim controller.*
- *There exists $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ s.t.*

$$B_\perp(AY + YA')B_\perp' < 0$$
$$(C')_\perp(XA + A'X)((C')_\perp)' < 0$$
$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \text{ and } \mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$$

1. Convex if $n_K \geq n_G$ since the rank constraint can be removed.
2. How do we get the controller parameters?
3. How to solve rank constrained problem?

KTH

Controller Reconstruction

- Suppose $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ satisfies (ii) in Theorem 2.
- By Schur complement $X - Y^{-1} = X_2 X_2'$ for some $X_2 \in \mathbf{R}^{n_G \times n_K}$
- Then $P = \begin{bmatrix} X & X_2 \\ X_2' & I \end{bmatrix} > 0$ (by Schur complements formula).
- $$\begin{bmatrix} X & X_2 \\ X_2' & I \end{bmatrix}^{-1} = \begin{bmatrix} Y & -YX_2 \\ -X_2'Y & I + X_2'YX_2 \end{bmatrix}$$
- We have found our Lyapunov matrix!
- Find controller parameters from LMI

$$P(\mathcal{A} + \mathcal{B}K\mathcal{C}) + (\mathcal{A} + \mathcal{B}K\mathcal{C})'P < 0$$

Heuristic for Rank Constrained Problems

A often used heuristic method: solve

$$\begin{aligned}&\min \quad \mathbf{tr}(X+Y) \quad \text{subj. to}\\ &B_\perp(AY+YA')B_\perp' < 0\\ &(C')_\perp(XA+A'X)((C')_\perp)' < 0\\ &X = X' > 0, \quad Y = Y' > 0, \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0\end{aligned}$$

Hopefully we get $\mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$. Of course, there is absolutely no guarantee at all.

r static output feedback problem:

Corollary 1. *There exists K such that $A+BKC$ is stable if and only if there exists $P,\ Q \in \mathcal{S}_+^{n_G \times n_G}$ such that*

$$\begin{aligned}&B_\perp(AQ+QA')B_\perp' < 0\\ &(C')_\perp(PA+A'P)((C')_\perp)' < 0\\ &\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0 \text{ and } \mathbf{rank} \begin{bmatrix} P & I \\ I & Q \end{bmatrix} \leq n_G\end{aligned}$$

- This is an NP hard problem
- The last condition is equivalent to $PQ = I$.
- Whether there exists a polynomial-time algorithm for finding static output feedback controller is still an open problem.

Design for Performance

Consider the following system

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u\\ z &= C_1 x + D_{11} w + D_{12} u\\ y &= C_2 x + D_{21} w\end{aligned}$$

- w disturbance signal
- z measured output

Controller:

$$\begin{aligned}\dot{x}_K &= A_K x_K + B_K y\\ u &= C_K x_K + D_K y\end{aligned} \qquad K := \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$$

Closed Loop System

With some algebraic manipulation, we can express the closed-loop system as

$$\begin{aligned}\dot{x}_{cl} &= A_{cl} x_{cl} + B_{cl} w\\ z &= C_{cl} x_{cl} + D_{cl} w\end{aligned}$$

where

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & D_0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ \mathcal{D}_{12} \end{bmatrix} K \begin{bmatrix} \mathcal{C} & \mathcal{D}_{21} \end{bmatrix}$$

$$\mathcal{A}_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_0 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

$$\mathcal{C}_0 = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \quad \mathcal{D}_0 = D_{11}$$

$$\mathcal{B} = \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix},$$

$$\mathcal{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \quad \mathcal{D}_{12} = \begin{bmatrix} 0 & D_{12} \end{bmatrix},$$

Let $K = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$. Then

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_0 & \mathcal{B}_0 \\ \mathcal{C}_0 & D_0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} \\ \mathcal{D}_{12} \end{bmatrix} K \begin{bmatrix} \mathcal{C} & \mathcal{D}_{21} \end{bmatrix}$$

Design Specifications

1. Closed loop system stable
2. Performance on channel $w \to z$

$$\int_0^\infty z(t)'Qz(t) + 2z(t)'Sw(t) + w(t)'Rw(t)\ dt \le 0$$

$$\Updownarrow$$

$$\int_{-\infty}^\infty \hat{z}(j\omega)^*Q\hat{z}(j\omega) + 2\mathbf{Re}(\hat{z}(j\omega)^*S\hat{w}(j\omega)) + \hat{w}(j\omega)^*R\hat{w}(j\omega)\ d\omega \le 0$$

$$\Updownarrow$$

$$\begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} < 0, \ \forall \omega \in [0, \infty]$$

- Special Case $\mathbf{H}_\infty$-synthesis

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Interpretation: Closed loop stable for all $\|\Delta\|_{\mathbf{H}_\infty} \le 1$

KYP Lemma

Lemma 3. *Assume $Q \geq 0$. Then the following are equivalent*

(i) G_{cl} is stable and

$$\begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} < 0, \quad \forall \omega \in [0, \infty]$$

(ii) There exists $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ such that

$$\begin{bmatrix} PA_{cl} + A_{cl}'P & PB_{cl} \\ B_{cl}'P & 0 \end{bmatrix} + \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix}' \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} C_{cl} & D_{cl} \\ 0 & I \end{bmatrix} < 0$$

Assume $Q > 0$. Then by the Schur's complement formula, (ii) is equivalent to

$$\begin{bmatrix} PA_{cl} + A_{cl}'P & PB_{cl} + C_{cl}'S & C_{cl}' \\ B_{cl}'P + S'C_{cl} & D_{cl}'S + S'D_{cl} + R & D_{cl}' \\ C_{cl} & D_{cl} & -Q^{-1} \end{bmatrix} < 0$$

The problem is to find P and K such that this inequality holds.

ıe Synthesis Problem:

Find $P \in \mathcal{S}_+^{(n_G+n_K)\times(n_G+n_K)}$ and $K = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix}$ such that

$$NKM' + MK'N' + H < 0$$

where

$$H = \begin{bmatrix} P\mathcal{A}_0 + \mathcal{A}_0'P & P\mathcal{B}_0 + \mathcal{C}_0'S & \mathcal{C}_0' \\ \mathcal{B}_0'P + S'\mathcal{C}_0 & \mathcal{D}_0'S + S'\mathcal{D}_0 + R & \mathcal{D}_0' \\ \mathcal{C}_0 & \mathcal{D}_0 & -Q^{-1} \end{bmatrix}$$

$$N = \begin{bmatrix} P\mathcal{B} \\ S'\mathcal{D}_{12} \\ \mathcal{D}_{12} \end{bmatrix}, \qquad M = \begin{bmatrix} \mathcal{C}' \\ \mathcal{D}_{21}' \\ 0 \end{bmatrix}$$

ıne elimination lemma gives the following necessary and sufficient conditions

$$N_\perp H N_\perp' < 0$$
$$M_\perp H M_\perp' < 0$$

where

$$N_\perp = \left[\begin{array}{cc|cc} V_1 & 0 & 0 & V_2 \\ 0 & 0 & I & -S' \end{array}\right] \left[\begin{array}{ccc} P^{-1} & 0 & 0 \\ \hline 0 & I & 0 \\ 0 & 0 & I \end{array}\right], \quad \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ D_{12} \end{bmatrix}_\perp$$

$$M_\perp = \begin{bmatrix} W_1 & 0 & W_2 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \begin{bmatrix} W_1 & W_2 \end{bmatrix} = \begin{bmatrix} C_2' \\ D_{21}' \end{bmatrix}_\perp$$

The orthogonal complements are as indicated since

$$N = \left[\begin{array}{c|cc} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \left[\begin{array}{cc} 0 & B_2 \\ I & 0 \\ \hline 0 & S'D_{12} \\ 0 & D_{12} \end{array}\right], \qquad M = \begin{bmatrix} 0 & C_2' \\ I & 0 \\ 0 & D_{21}' \\ 0 & 0 \end{bmatrix}$$

Next let $P = \begin{bmatrix} X & X_2 \\ X_2' & X_3 \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} Y & Y_2 \\ Y_2' & Y_3 \end{bmatrix}$. By Lemma 2 and some matrix manipulation of

$$N_\perp H N_\perp' < 0$$
$$M_\perp H M_\perp' < 0$$

we arrive the next theorem (it is a good exercise to fill in the details!).

Theorem 3. *The following are equivalent*

(i) G_{cl} *is stable and* $\begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} Q & S \\ S' & R \end{bmatrix} \begin{bmatrix} G_{cl}(j\omega) \\ I \end{bmatrix} < 0$, $\forall \omega \in [0, \infty]$

(ii) There exists $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ *such that*

$$\begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} AY + YA' & YC_1' & B_1 \\ C_1 Y & -Q^{-1} & D_{11} - Q^{-1}S' \\ B_1' & D_{11}' - SQ^{-1} & R - SQ^{-1}S' \end{bmatrix} \begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'X + XA & XB_1 + C_1'S & C_1' \\ B_1'X + S'C_1 & D_{11}'S + S'D_{11} + R & D_{11}' \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix} \begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \quad \textbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$$

where $\widetilde{N} := \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ and $\widetilde{M} := \begin{bmatrix} W_1 & W_2 \end{bmatrix}$

- Convex problem if $n_K \geq n_G$ since rank constraint disappears
- An important special case is $Q = I$, $R = -I$, and $S = 0$. This is the $\mathbf{H}_\infty$-synthesis problem. It gives

$$\|G_{cl}\|_{\mathbf{H}_\infty} < 1$$

Controller Reconstruction

- Suppose $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ satisfies (ii) in Theorem 3.
- By Schur complement $X - Y^{-1} = X_2 X_2'$ for some $X_2 \in \mathbf{R}^{n_G \times n_K}$
- Use $P = \begin{bmatrix} X & X_2 \\ X_2' & I \end{bmatrix} > 0$ (by Schur's lemma).
- Controller parameters from LMI

$$NKM' + MK'N' + H < 0$$

where H, N and M was as defined previously.

Theorem 4 ($\mathbf{H}_\infty$-synthesis: the case where $Q = I, S = 0, R = -I$). *The following are equivalent*

(i) $\|G\|_{\mathbf{H}_\infty} < 1$

(ii) *There exists* $X, Y \in \mathcal{S}_+^{n_G \times n_G}$ *such that*

$$\begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} AY + YA' & YC_1' & B_1 \\ C_1 Y & -I & D_{11} \\ B_1' & D_{11}' & -I \end{bmatrix} \begin{bmatrix} \widetilde{N} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix}' \begin{bmatrix} A'X + XA & XB_1 & C_1' \\ B_1' X & -I & D_{11}' \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} \widetilde{M} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \quad \mathbf{rank} \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \leq n_G + n_K$$