The Youla Parametrization

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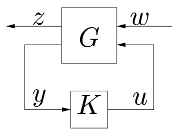
References

- Essentials of Robust Control, by Zhou and Doyle
- A Course in Robust Control Theory, by Dullerud and Paganini

Outline

- What is the Youla parametrization?
- State space formulas for linear fractional transformations (LFT) and the Redheffer starproduct
- Necessary and sufficient condition for output stabilizability
- The Youla parametrization

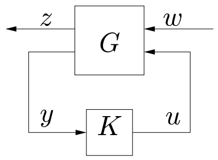
What is the Youla Parametrization?



- z regulated output
- y sensor output
- w exogenous input (disturbances)
- u control input

The Youla parametrization is a complete parametrization of all stabilizing proper LTI controllers.

LFT formulas



$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

The lower fractional transformation

$$\mathcal{S}_l(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

We have

$$\mathcal{S}_l(G,K) = \left[\begin{array}{cc|c} A + B_2D_KC_2 & B_2C_K & B_1 + B_2D_KD_{21} \\ B_KC_2 & A_K & B_KD_{21} \\ \hline C_1 + D_{12}D_KC_2 & D_{12}C_K & D_{11} + D_{12}D_KD_{21} \end{array}\right]$$

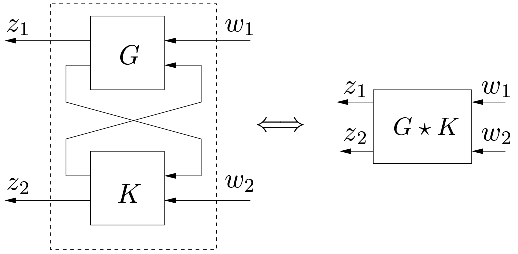
Proof:

$$\dot{x} = Ax + B_1w + B_2u = Ax + B_1w + B_2(C_Kx_K + D_K(C_2x + D_{21}w))$$

$$\dot{x}_K = A_Kx_K + B_K(C_2x + D_{21}w)$$

Note that $D_{22} \neq 0$ complicates the formulas a little

The Redheffer Star Product



$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \qquad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$G \star K := \mathcal{S}(G,K) = \begin{bmatrix} \mathcal{S}_l(G, K_{11}) & G_{12}(I - K_{11}G_{22})^{-1}K_{12} \\ K_{21}(I - G_{22}K_{11})^{-1}P_{21} & \mathcal{S}_u(K, G_{22}) \end{bmatrix}$$

State space formulas. If

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right]$$

Then

$$G \star K = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

$$\bar{A} = \begin{bmatrix} A + B_2 D_{K11} C_2 & B_2 C_{K1} \\ B_{K1} C_2 & A_K \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} B_1 + B_2 D_{K11} D_{21} & B_2 D_{K12} \\ B_{K1} D_{21} & B_{K2} \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C_1 + D_{12} D_{K11} C_2 & D_{12} C_{K1} \\ D_{K21} C_2 & C_{K2} \end{bmatrix}$$

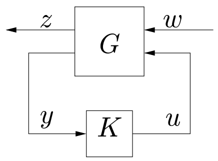
$$\bar{D} = \begin{bmatrix} D_{11} + D_{12} D_{K11} D_{21} & D_{12} D_{K12} \\ D_{K21} D_{21} & D_{K22} \end{bmatrix}$$

Proof:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ &= Ax + B_1 w + B_2(C_{K1} x_K + D_{K11}(C_2 x + D_{21} w) + D_{K12} \hat{w}) \\ &= (A + B_2 D_{K11} C_2)x + B_2 C_{K1} x_K + (B_1 + B_2 D_{K11} D_{21}) w + B_2 D_{K12} \hat{w} \\ \dot{x}_K &= A_K x_K + B_{K1} y + B_{K2} \hat{w} \\ &= A_K x_K + B_{K1}(C_2 x + D_{21} w) + B_{K2} \hat{w} \\ &= B_{K1} C_2 x + A_K x_K + B_{K1} D_{21} w + B_{K2} \hat{w} \end{aligned}$$

and similarly with the output equations.

Output Stabilizability



Theorem 1. *Let G be as before. There exists a proper internally stabilizing controller K iff (A, B_2) is stabilizable and (C_2, A) is detectable. If $A + B_2 F$ and $A + LC_2$ are Hurwitz then the following observer based controller*

$$K = \left[\begin{array}{c|c} A + B_2 F + LC_2 & -L \\ \hline F & 0 \end{array}\right] \tag{1}$$

gives an internally stable closed loop system.

Proof: The closed loop system $\mathcal{S}_l(G, K)$ has A_{cl} matrix

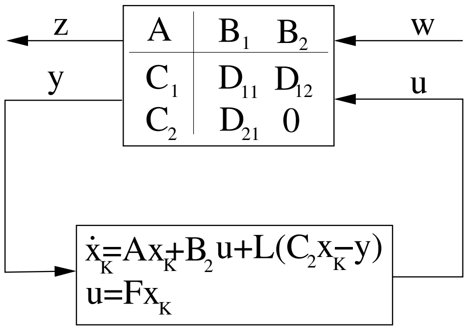
$$A_{cl} = \begin{bmatrix} A & B_2 F \\ -LC_2 & A + B_2 F + LC_2 \end{bmatrix}$$

Similarity transformation with $M = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$ gives

$$M^{-1} A_{cl} M = \begin{bmatrix} A + B_2 F & -B_2 F \\ 0 & A + LC_2 \end{bmatrix}$$

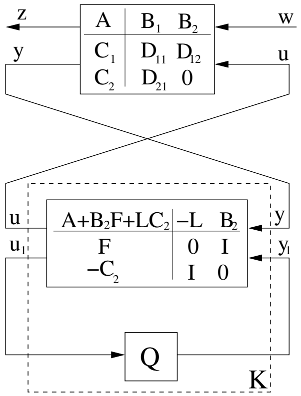
This proves sufficiency. The Neccesity follows because otherwise either (or both) $A + LC_2$ and $A + B_2 F$ cannot be made stable. A direct term D_k in the controller will not help.

The controller (1) is observer based.



To obtain the Youla parametrization we connect an arbitrary stable transfer function Q to the observer based controller in such a way that the transfer function from u to Q is zero.

The Youla Parametrization



The transfer function from u to u_1 is zero. Hence, the control $u = Fx_K + Qu_1$ is stabilizing because $u = Fx_K$ is a stabilizing observer based controller.

Theorem 2 (Youla Parametrization). *Let G be as before and*

$$J = \left[\begin{array}{c|cc} A + B_2F + LC_2 & -L & B_2 \\ \hline F & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

If (A, B_2) is stabilizable and (C_2, A) is detectable then we have

(i) Any internally stablizing proper controller has the form $K = \mathcal{S}_l(J, Q)$, where $Q \in \boldsymbol{RH}_\infty$.

(ii) The closed loop system becomes $T_{cl} = \mathcal{S}_l(T, Q)$, where

$$T = \left[\begin{array}{cc|cc} A & B_2F & B_1 & B_2 \\ -LC_2 & A + B_2F + LC_2 & -LD_{21} & B_2 \\ \hline C_1 & D_{12}F & D_{11} & D_{12} \\ C_2 & -C_2 & D_{21} & 0 \end{array}\right]$$

Remark 1. Introduction of new coordinates using $M = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix}$ gives (same realization as in the book by Zhou)

$$T = \left[\begin{array}{c|c} M^{-1}A_TM & M^{-1}B_T \\ \hline C_TM & D_T \end{array}\right]$$

$$= \left[\begin{array}{cc|cc} A + B_2F & -B_2F & B_1 & B_2 \\ 0 & A + LC_2 & B_1 + LD_{21} & 0 \\ \hline C_1 + D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right]$$

This shows that $T_{cl} = T_{11} + T_{12}QT_{21}$. Note that T_{11} is the observer based control system obtained in Theorem 1.

Proof: We have already proven the sufficiency for (i). To prove (ii) we use

$$\mathcal{S}_l(G,K) = \mathcal{S}_l(G, \mathcal{S}_l(J,Q)) = \mathcal{S}_l((G \star J), Q) = \mathcal{S}_l(T,Q)$$

since the star product formula gives $(G \star J) = T$.
To prove the necessity of (i) we let K be an arbitrary stabilizing controller. Consider the construction in the figure on the next slide.

Let $\hat{J} = \left[\begin{array}{c|cc} A & -L & B_2 \\ \hline -F & 0 & I \\ C_2 & I & 0 \end{array}\right]$. The proof follows since

- $Q = \mathcal{S}_l(\hat{J}, K) \in \mathbf{RH}_\infty$ (i.e. it is stable)
- $J \star \hat{J} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$