Characterization of Robust Stability of a Class of Interconnected Systems

Chung-Yao Kao [a,2], Ulf Jönsson [b,3] Hisaya Fujioka [c]

[a] *Department of Electrical and Electronic Engineering, University of Melbourne, Parkville 3010, Victoria, Australia*

[b] *Optimization and Systems Theory, Royal Institute of Technology, 10044 Stockholm, Sweden*

[c] *Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan*

Abstract

We consider robust stability analysis of a class of large scale interconnected systems. The individual subsystems may be different but they will be assumed to share a property that characterizes the interconnection matrix. The main contribution of the paper is to show that, for the case where the network interconnection matrix is normal, (robust) stability verification can be simplified to a low complexity problem of checking that the frequency response of the individual dynamics and the eigenvalues of the interconnection matrix can be mutually separated using a class of quadratic forms. Most interestingly, we show that this criterion provides a necessary and sufficient characterization of robust stability.

Key words: Networked Control Systems, Scalable Stability Criterion, Robust Stability, Graphical Test

1 Introduction

In this paper, robust stability analysis of a class of spatially interconnected systems is considered. Our main aim is to significantly reduce the computational complexity associated with robust stability analysis of such systems by exploiting the network structure. This goal is in line with the recent development of this field of research. The interconnection structures and decomposition techniques considered here are related to recent development on synchronization problems [17,12], consensus problems [13], vehicle formations [4], and other network problems with symmetry [2,5]. In these works, various techniques such as Nyquist criteria, Lyapunov theory, passivity, and dissipation theory are used for the analysis.

The class of interconnected systems considered in the paper consists of heterogeneous linear time invariant (LTI) and single-input-single-output (SISO) components interconnected over a communication network with a certain symmetry property. The network connection is primarily modelled by an interconnection matrix of complex value; however, the results presented in this paper can be generalized to encompass the case where dynamics of the link is taken into account.

Standard stability analysis of such interconnected systems will be computationally expensive, or even intractable, if the number of the components in the network is very big. This has motivated researchers to develop so-called scalable stability criteria. An early contribution was obtained by Vinnicombe in [16] where it is shown how structure can be explored to reduce the stability test to a separating hyperplane condition on the individual dynamics of the network. This result have been subsequently generalized [8,9] and applied to problems in congestion control for the Internet [10,15].

In [?] we showed that by making the hyperplane condition in [8] frequency dependent, it is possible to derive a necessary and sufficient stability characterization for heterogeneous LTI SISO systems interconnected over a symmetric network. In this paper we show that when the interconnection matrix is allowed to have a more general eigen-structure, it is still possible obtain a necessary and sufficient characterization for robust stability of the interconnected system. The new stability criterion has an appealing graphical interpretation which

[1] This paper was not presented at any IFAC conferences.
[2] Supported by the Australian research council.
[3] Supported by a grant from the Swedish Research Council (VR), the European Commission research project FP6-IST-511368 *Hybrid Control (HYCON)*

essentially says that a generalized Nyquist plot of the interconnected system should stay outside a polyhedron induced by the location where the eigenvalues of the interconnection matrix reside.

Notations

We let $\mathcal{A}$ denote the class of transfer functions obtained as the Laplace transform of the impulse response functions

$$h(t) = h_0(t) + \sum_{k=1}^{N} h_k \delta(t - t_k)$$

where $h_0 \in \mathbf{L}_1[0,\infty)$, $h_k \in \mathbf{R}$, $t_k \geq 0$ and $\delta(\cdot)$ is the Dirac function.

A linear system $\Delta \in \mathcal{A}$ is bounded and causal on $\mathbf{L}_2[0,\infty)$ and in particular we have $\|\Delta v\| \leq \sup_{\omega \in \mathbf{R}} |\Delta(j\omega)| \|v\|$, $\forall v \in \mathbf{L}_2[0,\infty)$, where $\|v\|^2 = \int_0^\infty |v(t)|^2 dt$.

The direct matrix sum is defined as

$$\oplus_{k=1}^{n} \Delta_k = \mathrm{diag}(\Delta_1, \ldots, \Delta_n).$$

For the case where $\Delta_k \in \mathcal{A}$ is a system, the later definition should be interpreted as

$$(\oplus_{k=1}^{n} \Delta_k)(v) := \begin{bmatrix} \Delta_1(v_1) \\ \vdots \\ \Delta_n(v_n) \end{bmatrix}$$

where $v = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T \in \mathbf{L}_2^n[0,\infty)$.

2 Problem Formulation and Preliminaries

We consider the class of systems in the following form

$$v = \Gamma w, \quad w = \Delta(v) + e \tag{1}$$

where $\Delta = \oplus_{k=1}^{n} \Delta_k$ and each $\Delta_k \in \mathcal{A}$. The disturbance $e \in \mathbf{L}_2^n[0,\infty)$ models the initial condition and the interconnection Γ may generally be modeled as a transfer function in $\mathcal{A}^{n\times n}$ in order to include time delays and bandwidth constraints. To keep the discussion as simple as possible, we will assume $\Gamma \in \mathbf{C}^{n\times n}$. However, the results presented in this paper can be easily generalized to cover the case where $\Gamma \in \mathcal{A}^{n\times n}$. The system (1) is assumed to be well-posed in the sense that there always exists a solution in extended $\mathbf{L}_2$-space, see [3].

The system in (1) may be viewed as a set of heterogeneous plants interconnected over a network defined by Γ. We will consider the case where Γ is unitarily diagonalizable, i.e., it can be expressed as $U\mathcal{D}U^*$ where U is a unitary matrix and Λ is a diagonal matrix whose diagonal entries consisting of the eigenvalues of Γ. Interconnected systems where the interconnection matrices are normal do appear in some engineering applications. For example, such systems arise in modelling vehicle platoons, Internet congestion control, and feedback control of biochemical pathways. In Section 4, we will use the vehicle platoon as an example to illustrate the analysis methodology proposed in this paper. The necessary and sufficient condition for unitary diagonalizability is normality: a matrix Γ can be unitarily diagonalized if and only if it is a normal matrix, i.e. it satisfies $\Gamma\Gamma^* = \Gamma^*\Gamma$.

We are interested in deriving computationally tractable conditions for (standard) finite-gain input-output stability of networked systems in the form of (1). With this we mean that there exists a scalar $c > 0$ such that $\|w\| + \|v\| \leq c\|e\|$, for all input disturbances $e \in \mathbf{L}_2^n[0,\infty)$.

Computational tractability is obtained by exploiting the unitary diagonalization of Γ to decompose the stability criterion into a scalar criterion that each individual dynamics must satisfy. This not only implies a significant reduction of computational complexity in numerical tests of stability but it also has an associated graphical test by which the stability condition can be easily visualized. The price paid for these benefits is that the stability test may be conservative in general; however, as we will show in our main results, the test provides a necessary and sufficient characterization of *robust* stability for a set of network interconnections.

We will use quadratic inequalities to characterize the eigenvalue location of Γ. To this end, consider a subset of $\mathbf{C}$

$$\begin{aligned} \Lambda := \{\lambda : |\lambda|^2 + 2\pi_1 \mathrm{Re}\,\lambda - 2\pi_2 \mathrm{Im}\,\lambda \leq 0, \\ \forall \pi_1 \in [\underline{\gamma}_1, \bar{\gamma}_1],\ \pi_2 \in [\underline{\gamma}_2, \bar{\gamma}_2]\} \end{aligned} \tag{2}$$

and a set of normal matrices

$$\begin{aligned} \mathcal{G} = \{\Gamma : \Gamma \in \mathbf{C}^{n\times n};\ \Gamma\Gamma^* = \Gamma^*\Gamma \text{ and all } \lambda_i \in \mathrm{eig}(\Gamma) \\ \text{satisfy } \lambda_i \in \Lambda\}. \end{aligned}$$

The eigenvalue locations defined by Λ is the intersection of all circles in the complex plane which have centers $-(\pi_1 + i\pi_2)$ and radius $\sqrt{\pi_1^2 + \pi_2^2}$, see Fig 1 for an illustration. Note that the origin is on the boundary of Λ. It is a standing assumption that the convex set created by the intersection of all these circles contains non-zero elements.

Before proceeding to the main result, we give some examples where the form of interconnection matrices, their

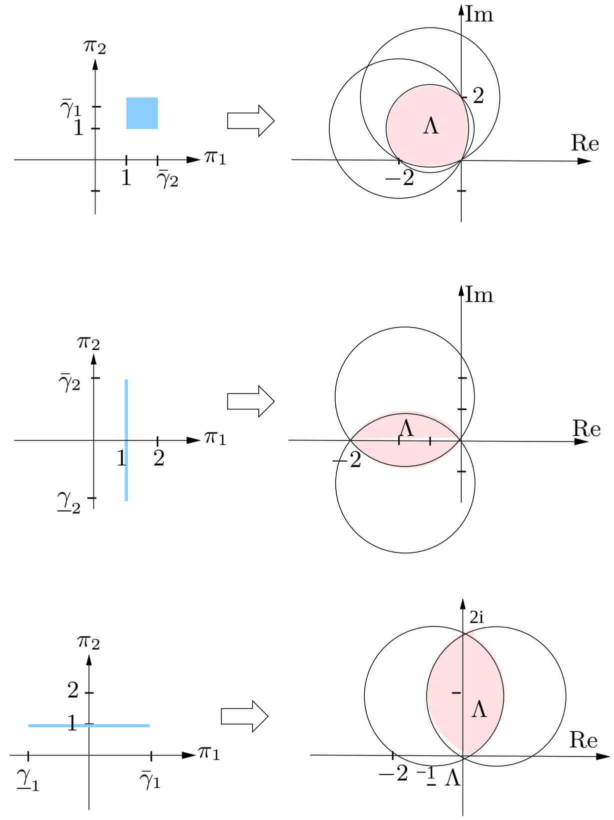

Fig. 1. Illustration of Λ. Λ is characterized by the ranges of π_1 and π_2. In the first example, the rectangular area in $\pi_1 - \pi_2$ space produces the shaded region in the complex plane. In the next two examples only one of the two parameters is varied.

eigenvalue locations, and how to characterize these locations using (2) are shown. The first example outlines a number of cases of practical interest. The next two examples show that loop transformations sometimes must be used to obtain the eigenvalue distribution in (2).

(i) Consider system (1) where the feedback is generated by

$$v_k = -\frac{1}{d_k}\sum_{l\neq k} a_{kl}(w_k - w_l)$$

where $d_k = \sum_{l\neq k} a_{kl}$. Then the interconnection matrix Γ is equal to $-I + D^{-1}A$ where matrix D is equal to $\mathrm{diag}(d_1, \cdots, d_n)$ and matrix A has the form $[A_{k,l}] = a_{kl}$. Note that a_{kl} is zero if nodes k and l are not connected.

It can be shown using the Geršhgorin's theorem that all eigenvalues of Γ lie in a disk of radius 1 centered at the point $-1 + 0j$. Hence, Λ can be characterized as

$$\Lambda := \{\lambda : |\lambda|^2 + 2\mathrm{Re}\,\lambda \le 0\}$$

Additional conditions on the coefficients a_{kl} are needed to ensure that Γ is normal. Three cases of interest are

(a) $a_{kl} = c_{(l-k+1) \mathrm{mod}\, n}$ in which case Γ is a circulant matrix. Here $c_1, \ldots, c_n$ are real numbers. They determine the eigenvalues via the discrete Fourier transform

$$\lambda_l = \sum_{k=1}^{n} c_k e^{j2\pi l(k-1)/n}.$$

(b) $a_{lk} = \bar{a}_{kl}$ in which case Γ is Hermitian and the eigenvalues are located in the interval $[-2, 0]$. This corresponds to the characterization in (2) with $\bar\gamma_1 = \underline{\gamma}_1 = 1$, $\bar\gamma_2 = \infty$ and $\underline{\gamma}_2 = -\infty$.

(c) $a_{lk} = -\bar{a}_{kl}$ in which case Γ is skew-Hermitian and the eigenvalues are located in the interval $[-2j, 0]$ on the imaginary axis. This corresponds to the characterization in (2) with $\bar\gamma_2 = \underline{\gamma}_2 = 1$, $\bar\gamma_1 = \infty$ and $\underline{\gamma}_1 = -\infty$.

(ii) bidirectional circular graph with weighted links: in this case, the interconnection matrix has the form

$$\Gamma = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & \ldots & 1 \\ 1 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & 0 & 1 & \ldots & \ldots & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & \ldots & \ldots & 0 & 1 & 0 & 1 \\ 1 & \ldots & \ldots & 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be shown that all eigenvalues of this Γ are real and lie in $[-1, 1]$. By an loop transformation where one replaces Γ by $\Gamma - I$ and Δ_k by $\frac{\Delta_k}{1-\Delta_k}$, one arrives an equivalent system where the new interconnection matrix has all its eigenvalues lying in $[-2, 0]$. This interval can be characterized as $\Lambda := \{\lambda : |\lambda|^2 + 2\mathrm{Re}\,\lambda - 2\pi_2 \mathrm{Im}\,\lambda \le 0,\ \forall \pi_2 \in (-\infty, \infty)\}$.

(iii) bidirectional circular graph with weighted links and negative feedback: in this case, the interconnection matrix has the form

$$\Gamma = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & \ldots & -1 \\ -1 & 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 0 & 1 & \ldots & \ldots & 0 \\ & & \ddots & \ddots & \ddots & & \\ 0 & \ldots & \ldots & 0 & -1 & 0 & 1 \\ 1 & \ldots & \ldots & 0 & 0 & -1 & 0 \end{bmatrix}$$

All eigenvalues of this Γ are purely imaginary and lie in the interval $[-j, j]$. Similarly, by an loop transformation where one replaces Γ by $\Gamma - j \cdot I$ and Δ_k by $\frac{\Delta_k}{j-\Delta_k}$, one arrives an equivalent system where the new interconnection matrix has all its eigenvalues lying in $[-2j, 0]$. This interval can be characterized as $\Lambda := \{\lambda : |\lambda|^2 + 2\pi_1 \mathrm{Re}\,\lambda - 2\mathrm{Im}\,\lambda \leq 0,\ \forall \pi_1 \in (-\infty, \infty)\}$

3 A Scalable Stability Criterion

The next results provides a low complexity stability test for (1) when Γ belongs to $\mathcal{G}$.

Proposition 1 *Consider the interconnected system (1), where each $\Delta_k \in \mathcal{A}$ and the interconnection matrix $\Gamma \in \mathcal{G}$. Then the system is stable if there exist scalar functions $\pi_1(\omega)$ and $\pi_2(\omega)$, where $\pi_1(\omega) \in [\underline{\gamma}_1, \bar{\gamma}_1]$ and $\pi_2(\omega) \in [\underline{\gamma}_2, \bar{\gamma}_2]$, such that for $k = 1, ..., n$,*

$$1 + 2\pi_1(\omega) Re\,\Delta_k(j\omega) + 2\pi_2(\omega) Im\,\Delta_k(j\omega) > 0, \qquad (3)$$

for all $\omega \in [0, \infty]$.

PROOF. We will use integral quadratic constraints (IQC) along the lines of [11] to prove the result. We define the multiplier

$$\Pi(\omega) = \begin{bmatrix} I_n & \pi_{12}(\omega) I_n \\ \bar{\pi}_{12}(\omega) I_n & 0_n \end{bmatrix}.$$

where $\pi_{12}(\omega) = \pi_1(\omega) + i\pi_2(\omega)$ and $\bar{\pi}_{12}(\omega) = \pi_1(\omega) - i\pi_2(\omega)$. We notice that $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$ so stability follows if we verify that Γ and Δ satisfies two complementary IQCs whereof one must be strict. Since the operators are time-invariant, the IQCs reduce to frequency wise quadratic inequalities:

$$\begin{bmatrix} \Gamma \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} \Gamma \\ I \end{bmatrix} \leq 0, \quad \forall\ \omega \in [0, \infty]$$

$$\begin{bmatrix} I \\ \Delta(\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \Delta(\omega) \end{bmatrix} > 0, \quad \forall\ \omega \in [0, \infty]$$

Given the assumption that Γ is normal, we have

$$\begin{bmatrix} \Gamma \\ I \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} \Gamma \\ I \end{bmatrix} = U^* \left(|\mathcal{D}|^2 + \bar{\mathcal{D}} \bar{\pi}_{12}(\omega) + \mathcal{D} \pi_{12}(\omega) \right) U$$

which is non-positive since each diagonal entry in the middle matrix evaluates to

$$|\lambda_i|^2 + 2\pi_1(\omega) \mathrm{Re}\,\lambda_i - 2\pi_2(\omega) \mathrm{Im}\,\lambda_i \leq 0 \qquad (4)$$

which is non-positive since $\Gamma \in \mathcal{G}$. Finally

$$\begin{bmatrix} I \\ \Delta(\omega) \end{bmatrix}^* \Pi(\omega) \begin{bmatrix} I \\ \Delta(\omega) \end{bmatrix}$$
$$= \mathrm{diag}(1 + 2\pi_1(\omega) \mathrm{Re}\,\Delta_k(\omega) + 2\pi_2(\omega) \mathrm{Im}\,\Delta_k(\omega))$$

which is strictly positive by (3).

The significance of condition (3) lies in the modest computational complexity required for verifying the inequality numerically. The corresponding non-decomposed stability criterion involves a transfer function inequality of dimension $n \times n$, which should be compared with the n scalar valued inequalities in (3). Hence the computational complexity of verifying condition (3) grows *linearly* in n (after finding all eigenvalues of Γ, which has complexity $\mathrm{O}(n^3)$), while the complexity of verifying the non-decomposed condition in general grows as $\mathrm{O}(n^\beta)$, where β is between 4.5 to 6.5, see [**?**].

By exploring that all Δ_k are LTI and SISO, we are able to develop a simple graphical test for verifying robust stability of system (1). The rest of this section is devoted to this development.

3.1 A Graphical Test for Robust Stability

Note that in (3), the same function $\pi_i(\omega)$, $i = 1, 2$, appears in all n inequalities. Suppose that one has already verified all eigenvalues of Γ obey (4). Then checking stability of the system boils down to

$$\text{Find } \pi_1(\omega) \in [\underline{\gamma}_1, \bar{\gamma}_1] \text{ and } \pi_2(\omega) \in [\underline{\gamma}_2, \bar{\gamma}_2], \text{ such that inequality (3) holds for all } \omega \in [0, \infty]. \qquad (5)$$

Problem (5) has no solution if and only if there exists some $\omega_0 \in [0, \infty]$ such that no real numbers $\pi_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$ and $\pi_2 \in [\underline{\gamma}_2, \bar{\gamma}_2]$ makes $1 + 2\pi_1 \mathrm{Re}\,\Delta_k(j\omega_0) + 2\pi_2 \mathrm{Im}\,\Delta_k(j\omega_0)$ strictly positive for all $k = 1, \ldots, n$. This is equivalent to that the following two convex sets

$$C_1 = \{x \in \mathbf{R}^n : x_k = 1 + 2\pi_1 \mathrm{Re}\,\Delta_k(j\omega_0) + 2\pi_2 \mathrm{Im}\,\Delta_k(j\omega_0);\ \pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i],\ i = 1, 2\}$$
$$C_2 = \{x \in \mathbf{R}^n : x_k > 0\}$$

are disjoint. Hence, by the separating hyperplane theorem, one has an equivalent condition: there exists a nonzero $z \in \mathbf{R}^n$ such that $z_k \geq 0$ and $z^T x \leq 0$ for all $x \in C_1$. In other words,

$$\sum_{k=1}^{n} z_k (1 + 2\pi_1 \mathrm{Re}\,\Delta_k(j\omega_0) + 2\pi_2 \mathrm{Im}\,\Delta_k(j\omega_0)) \leq 0 \quad (6)$$

for all $\pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, 2$. Note that it is no restriction to assume that $\sum_{k=1}^{n} z_k = 1$. Hence, condition (6) is equivalent to

$$\pi_1 \sum_{k=1}^{n} z_k \mathrm{Re}\,\Delta_k(j\omega_0) + \pi_2 \sum_{k=1}^{n} z_k \mathrm{Im}\,\Delta_k(j\omega_0) \leq -0.5 \tag{7}$$

for all $\pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, 2$. Condition (7) has a geometrical interpretation: consider an ω-parameterized set

$$\mathcal{S}(\omega) = \{(\mathrm{Re}\,\Delta_k(j\omega), \mathrm{Im}\,\Delta_k(j\omega)) : k = 1, \cdots, n\}.$$

That condition (7) holds means the convex hull of set $\mathcal{S}(\omega_0)$ (denoted as $\mathrm{co}(\mathcal{S}(\omega_0))$) *intersects* the polyhedron

$$\mathcal{P} = \bigcap_{\pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i], i=1,2} \{(x_1, x_2) \mid \pi_1 \cdot x_1 + \pi_2 \cdot x_2 \leq -0.5\} \tag{8}$$

This in turn implies that problem (5) can be solved if and only if $\mathrm{co}(\mathcal{S}(\omega))$ does *not* intersect the polyhedron $\mathcal{P}$ for all ω. On the other hand, the condition $\Gamma \in \mathcal{G}$ requires all eigenvalues of Γ to be in the set Λ. This condition implies that the set

$$\mathcal{L} := \{(\mathrm{Re}\,(1/\lambda_k), \mathrm{Im}\,(1/\lambda_k)) \mid \lambda_k \in \mathrm{eig}(\Gamma), \lambda_k \neq 0\},$$

induced by the non-zero eigenvalues of Γ, must be inside $\mathcal{P}$.

This gives a graphical test for stability of the interconnected system (1) where the connection matrix Γ is unitarily diagonalizable and Δ_k are stable, single-input-single-output, linear time invariant operators: such a system is stable if the set $\mathcal{L}$ induced by the non-zero eigenvalues of the connection matrix Γ belongs to the polyhedron $\mathcal{P}$, and the convex hull $\mathrm{co}(\mathcal{S}(\omega))$ does not intersect $\mathcal{P}$ for all ω. If one views the path of $\mathrm{co}(\mathcal{S}(\omega))$ as a generalized Nyquist plot of the interconnected system, then the condition basically says that, for the interconnected system to be stable, the Nyquist plot must stay outside the polytope which contains $\mathcal{L}$, see Fig 2 for an illustration.

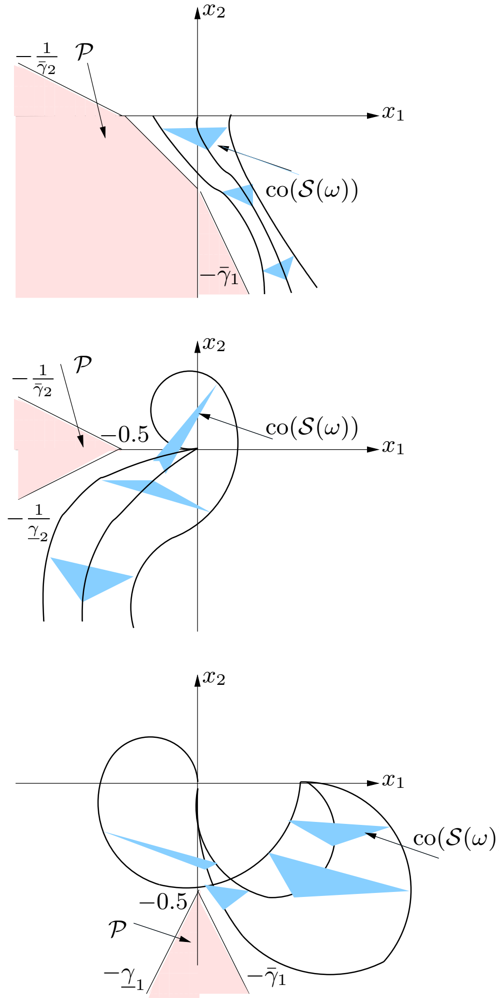

Fig. 2. Illustration of the graphical test for stability. The system is stable if the convex hull $\mathrm{co}(\mathcal{S}(\omega))$ does not intersect the polyhedron $\mathcal{P}$ for all ω. The three cases corresponds to the three cases in Fig. 4.

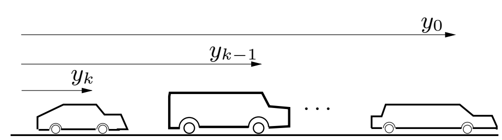

Fig. 3. Heterogeneous vehicle platoon.

4 Example: Heterogeneous Vehicle Platoon

We consider a heterogeneous vehicle platoon with a bi-directional control scheme

$$\begin{aligned} u_k &= C_k(e_k - e_{k+1}), \quad k = 1, \ldots, n-1 \\ u_n &= C_n e_n \\ e_k &= y_{k-1} - y_k - \delta \end{aligned}$$

where δ is the desired relative spacing between the vehicles. The first vehicle with index 0 merely serves as the leader and is not using information from the others, see Figure 3.

The dynamics of each vehicle is assumed to be of the form $H_k(s) = \frac{1}{s^2} G_k(s)$, where G_k is a stable transfer function with $G_k(0) \neq 0$. Normally the vehicles in the platoon are assumed to have the same dynamics [14,1] while some recent works have considered the extension to the heterogeneous case [6,7]. We here illustrate how our framework can be applied to this situation.

We assume each vehicle is initialized at rest (in particular zero velocity) with an initial position $y_k(0)$. This means that each vehicle position evolves according to

$$y_k(t) = (H_k u_k)(t) + y_k(0), \quad t \geq 0.$$

The complete feedback system has the equation

$$y = HC(\Gamma y + x) + y(0)\theta \tag{9}$$

where $\theta(t)$ is the unit step function and

$$\begin{aligned} H &= \text{diag}(H_1, \ldots, H_n) \\ C &= 2\text{diag}(C_1, \ldots, C_n) \\ \Gamma &= \frac{1}{2}\begin{bmatrix} -2 & 1 & 0 & 0 & \ldots & \ldots & 0 \\ 1 & -2 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 1 & -2 & 1 & \ldots & \ldots & 0 \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & \\ 0 & \ldots & \ldots & 0 & 1 & -2 & 1 \\ 0 & \ldots & \ldots & 0 & 0 & 1 & -1 \end{bmatrix} \\ x(t) &= y_0(t)\mathbf{e}_1 - \delta \mathbf{e}_n \theta(t) \end{aligned}$$

and where

$$y(t) = \begin{bmatrix} y_1(t) & \ldots & y_n(t) \end{bmatrix}^T, \quad y(0) = \begin{bmatrix} y_1(0) & \ldots & y_n(0) \end{bmatrix}^T$$
$$\mathbf{e}_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}^T, \quad \mathbf{e}_n = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix}^T$$

The matrix Γ is symmetric but the $H_k C_k$ are not bounded operators due to the poles at the origin. However, we will be able to transform the system such that Proposition 1 can be applied. To this end, we first note that the eigenvalues of Γ can be bounded as[4] ($\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$)

$$\begin{aligned} -2 \leq \lambda_1(\Gamma) &\leq -2\sin(\pi(n-1)/(2n)) \\ -2\sin(\pi/(2n))^2 \leq \lambda_n(\Gamma) &\leq -\frac{1}{n + n^2/2}. \end{aligned}$$

Since Γ is negative definite, we can thus make a loop transformation and turn (9) into an equivalent system

$$y = \Delta(\tilde{\Gamma} y + x - \mu y(0)\theta) + y(0)\theta$$

where

$$\begin{aligned} \widehat{\Gamma} &= \Gamma + \mu I, \quad \mu = -\lambda_n(\Gamma) \\ \Delta &= \text{diag}(\Delta_1, \ldots, \Delta_n), \quad \Delta_k = \frac{2H_k C_k}{1 + 2\mu H_k C_k}. \end{aligned}$$

[4] These bounds can be derived as in [1].

We assume each Δ_k is stable. This is a reasonable assumption if, for example, appropriately tuned PD controllers $C_k(s) = k_1 + k_2 s$ are used and if the dynamics of the G_k are "modest" perturbations of the identity. Note that eig $\widehat{\Gamma} \in [-2, 0]$; this interval can be characterized by

$$|\lambda|^2 + 2\text{Re}\,\lambda - 2\pi_2 \text{Im}\,\lambda \leq 0$$

for all $\pi_2 \in (-\infty, \infty)$. Then the stability criterion in Proposition 1 becomes: there exists a real valued function $\pi_2(\omega)$ such that

$$1 + 2\text{Re}\,\Delta_k(j\omega) + 2\pi_2(\omega)\text{Im}\,\Delta_k(j\omega) > 0, \quad \forall \omega \in [0, \infty]$$

for $k = 1, \ldots, n$. A graphical illustration of this condition is obtained by letting $\underline{\gamma}_2 = -\bar{\gamma}_2 = -\gamma$ in the second part of Fig 2 and let $\gamma \to \infty$, i.e. co$(\mathcal{S}(\omega))$ must avoid the interval $(-\infty, -0.5]$ on the real axis.

5 Necessity of the Graphical Test for Stability

For a given interconnection matrix Γ and a given set of transfer functions Δ_k, $k = 1, \ldots, N$, the stability criterion stated in Proposition 1 and the corresponding graphical test will generally be conservative. Indeed, by scaling the multiplier (i.e., $\Pi(\omega)$), we would generally obtain a less conservative stability criterion. However, in this case the verification of the stability criterion would be computationally expensive, because scaling the multiplier would result in combination of components in $\Pi(\omega)$ which turns condition (4) into one big operator inequality. Interestingly, however, it turns out that our low complexity graphical test provides a *necessary* characterization (in the sense which we will clarify shortly) of robust stability. To facilitate the development, let us consider the following set

$$\begin{aligned} \mathcal{D} = \{\Delta : \Delta = \oplus_{k=1}^n \Delta_k;\ &\text{each } \Delta_k \text{ is stable, LTI, and} \\ &\text{satisfies (3), where } \pi_i(\omega) \in [\underline{\gamma}_i, \bar{\gamma}_i], \text{ i=1,2.}\}. \end{aligned}$$

Theorem 1 *We have the following two necessary and sufficient conditions for stability of the interconnected system (1)*

(i) Let the connection matrix Γ be a given normal matrix. The interconnected system (1) is input-output stable for any $\Delta \in \mathcal{D}$ if and only if $\Gamma \in \mathcal{G}$.
(ii) Let $\Delta = \oplus_{k=1}^n \Delta_k$ where Δ_k are given stable LTI operators. The interconnected system (1) is input-output stable for any $\Gamma \in \mathcal{G}$ if and only if $\Delta \in \mathcal{D}$.

Remark 1 *In practice, a complex-valued entry of the network connection matrix may be viewed as a dynamical link (a communication channel with time-delay or limited bandwidth) transmitting signals of a particular frequency. We remark that all proofs in the paper hold even when the connection matrix Γ is frequency dependent, as long as $\Gamma(\omega)$ can be unitarily diagonalized for all ω.*

The sufficiency of statements (i) and (ii) of Theorem 1 follows from Proposition 1. The main idea behind the proof of necessity of the two statements is to explicitly construct examples which show that instability of the interconnected system could occur if the conditions are violated.

5.1 Proof of Necessity in Theorem 1

Proof of necessity for statement (i)*:* Suppose $\Gamma \notin \mathcal{G}$. Then at least one of the eigenvalues of Γ, denoted as λ_0, satisfies

$$|\lambda_0|^2 + 2\pi_1 \mathrm{Re}\,\lambda_0 - 2\pi_2 \mathrm{Im}\,\lambda_0 > 0, \tag{10}$$

for some $\pi_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$ and $\pi_2 \in [\underline{\gamma}_2, \bar{\gamma}_2]$. Note that if λ_0 satisfies (10), then $\lambda_0 \neq 0$. Furthermore, it can be verified that λ_0^{-1} satisfies

$$1 + 2\pi_1 \mathrm{Re}\,\lambda_0^{-1} + 2\pi_2 \mathrm{Im}\,\lambda_0^{-1} > 0.$$

Indeed, by noticing that $\mathrm{Re}\,\lambda_0^{-1} = \mathrm{Re}\,\lambda_0/|\lambda_0|^2$, and $\mathrm{Im}\,\lambda_0^{-1} = -\mathrm{Im}\,\lambda_0/|\lambda_0|^2$, the above inequality can be easily verified. This implies that, among those stable LTI operators which satisfy condition (3), it is possible to find one operator $\Delta_0(j\omega)$ which takes value λ_0^{-1} at certain frequency, say ω_0.

Let $\Delta \in \mathcal{A}^{n\times n} := \det(\Delta_0, \cdots, \Delta_0)$. Clearly, $\Delta \in \mathcal{D}$ and $\Delta(j\omega_0) = \lambda_0^{-1} \cdot I_n$. Let v_0 be the eigenvector of Γ corresponding to the eigenvalue λ_0. Then $\Gamma\Delta(j\omega_0)v_0 = v_0$ so the Nyquist locus of $\Gamma\Delta$ crosses the critical point $s = 1$ and the system is at best neutrally stable and not input-output stable.

Proof of necessity for statement (ii)*:* Note that (6) holds if and only if the following linear program (LP) can be solved and the optimal objective value is non-positive:

$$\min\ 1 + 2(\bar{\gamma}_1 \cdot z_{n+1} - \underline{\gamma}_1 \cdot z_{n+2} + \bar{\gamma}_2 \cdot z_{n+3} - \underline{\gamma}_2 \cdot z_{n+4})$$

$$\text{subject to} \quad \begin{cases} \sum_{k=1}^{n} z_k \mathrm{Re}\,\Delta_k(j\omega_0) = z_{n+1} - z_{n+2} \\ \sum_{k=1}^{n} z_k \mathrm{Im}\,\Delta_k(j\omega_0) = z_{n+3} - z_{n+4} \\ \sum_{k=1}^{n} z_k = 1 \\ z_k \geq 0, k = 1, \cdots n+4 \end{cases}$$

The linear program is of the standard form. It is well known that for such a linear program, either there exists an optimal solution with at most three nonzero variables or the problem is infeasible. Hence, if $\Delta \notin \mathcal{D}$, then there exists $\omega_0 \in [0, \infty]$ such that the LP is feasible, the optimal objective of the LP is less than or equal to zero. We then have the following possible scenarios for the optimal solution z^{op}:

(A) Only one z_k^{op} is nonzero, where it could be that
  (1) z_k^{op} is from $\mathcal{Z}_1 := \{z_1^{op}, \cdots, z_n^{op}\}$, or
  (2) z_k^{op} is from $\mathcal{Z}_2 := \{z_{n+1}^{op}, \cdots, z_{n+4}^{op}\}$.
(B) Two z_k are nonzero, where it could be that
  (1) one z_k^{op} is from $\mathcal{Z}_1$ and one z_k^{op} is from $\mathcal{Z}_2$, or
  (2) both z_k^{op} are from $\mathcal{Z}_1$, or
  (3) both z_k^{op} are from $\mathcal{Z}_2$.
(C) Three z_k^{op} are nonzero, where it could be that
  (1) one z_k^{op} is from $\mathcal{Z}_1$ and two z_k^{op} are from $\mathcal{Z}_2$, or
  (2) two z_k^{op} are from $\mathcal{Z}_1$ and one z_k^{op} is from $\mathcal{Z}_2$, or
  (3) all three z_k^{op} are from $\mathcal{Z}_1$, or
  (4) all three z_k^{op} are from $\mathcal{Z}_2$.

The condition $\sum_{k=1}^{n} z_k^{op} = 1$ implies that at least one element from $\mathcal{Z}_1$ must be non-zero. Hence scenarios (A2), (B3), and (C4) are invalid. Furthermore, scenarios (A1), (B2), and (C3) are also invalid, because in these cases, the optimal objective of the LP is 1, which is not less than or equal to 0.

Without loss of generality, let z_1^{op} to be nonzero in the cases (B1) and (C1), let z_1^{op} and z_2^{op} to be nonzero in the case (C2). Following the setup of the linear program and in particular $z_k \geq 0$, one can verify that [5]

(I) For the cases (B1) and (C1), we have

$$1 + 2\pi_1 \mathrm{Re}\,\Delta_1 + 2\pi_2 \mathrm{Im}\,\Delta_1 \leq 0 \tag{11}$$

for all $\pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, 2$.

(II) For the case (C2), either we have

$$\begin{aligned} &z_1^{op}\mathrm{Im}\,\Delta_1 + z_2^{op}\mathrm{Im}\,\Delta_2 = 0, \\ &1 + 2\pi_1(z_1^{op}\mathrm{Re}\,\Delta_1 + z_2^{op}\mathrm{Re}\,\Delta_2) \leq 0 \end{aligned} \tag{12}$$

for all $\pi_1 \in [\underline{\gamma}_1, \bar{\gamma}_1]$, or we have

$$\begin{aligned} &z_1^{op}\mathrm{Re}\,\Delta_1 + z_2^{op}\mathrm{Re}\,\Delta_2 = 0, \\ &1 + 2\pi_2(z_1^{op}\mathrm{Im}\,\Delta_1 + z_2^{op}\mathrm{Im}\,\Delta_2) \leq 0 \end{aligned} \tag{13}$$

for all $\pi_2 \in [\underline{\gamma}_2, \bar{\gamma}_2]$.

Figure 4 illustrates scenarios (11), (12), and (13). In the following, we discuss separately for each case how a connection matrix which destabilizes the interconnected system can be constructed.

[5] In the following, we will slightly abuse the notation by writing Δ_i for $\Delta_i(j\omega_0)$, for the sake of saving space.

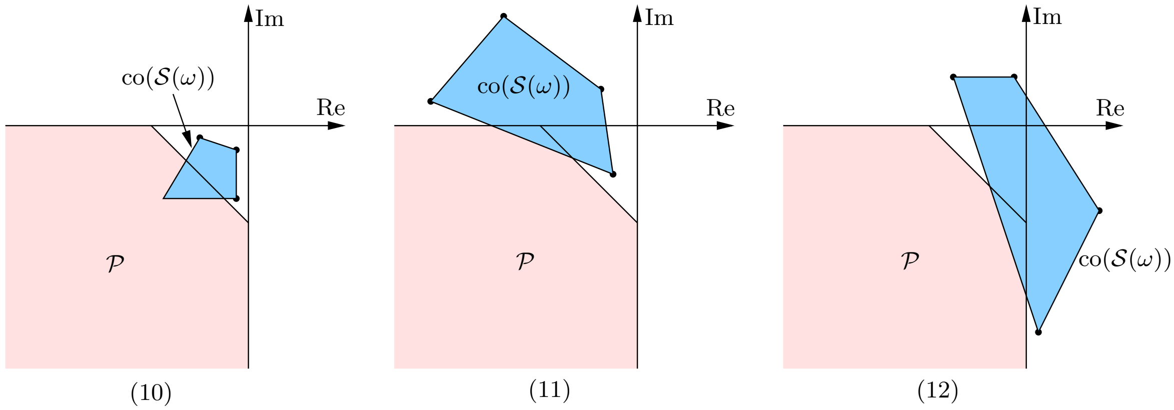

Fig. 4. Illustration of the cone condition. The polytope $\mathcal{P}$ corresponds to the first case in Fig. 2 when $\bar{\gamma}_1 = \bar{\gamma}_2 = \infty$. Left: a vertex of co($\mathcal{S}(j\omega_0)$) is inside polyhedron $\mathcal{P}$ but none of the edges of co($\mathcal{S}(j\omega_0)$) intersects the real or imaginary axes. This illustrates scenario (11). Middle: an edge of co($\mathcal{S}(j\omega_0)$) intersects the real axis. This illustrates scenario (12). Right: an edge of co($\mathcal{S}(j\omega_0)$) intersects the imaginary axis. This illustrates scenario (13).

For case (I)

We find Δ_1 satisfying inequality (11). Note this implies that $\Delta_1 \neq 0$. Let us consider the connection matrix $\Gamma = \delta \cdot I_n$, where $\delta = \Delta_1^{-1}$. Note that

$$\begin{aligned}&|\delta|^2 + 2\pi_1 \mathrm{Re}\,\delta - 2\pi_2 \mathrm{Im}\,\delta\\ &= \frac{1 + 2\pi_1 \mathrm{Re}\,\Delta_1 + 2\pi_2 \mathrm{Im}\,\Delta_1}{|\Delta_1|^2}.\end{aligned}$$

Hence, δ satisfies $|\delta|^2 + 2\pi_1 \mathrm{Re}\,\delta - 2\pi_2 \mathrm{Im}\,\delta \leq 0$ for all $\pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, 2$. Thus $\Gamma \in \mathcal{G}$. Now, $\Gamma\Delta(j\omega_0)$ obviously has an eigenvalue at 1 which implies that the system is at best neutrally stable and not input-output stable.

For case (II)

There are two possible scenarios (12) and (13). We first show how to construct de-stabilizing interconnection matrix when (12) holds. Construction of de-stabilizing interconnection matrix when (13) holds is very similar to that of (12). We will not pursue the full detail of the proof for scenario (13) but only comment on the difference from the case where (12) holds. Note that (12) holds implies that zero cannot belong to the interval where π_1 resides; therefore, either $0 < \underline{\gamma}_1 \leq \bar{\gamma}_1$ or $\underline{\gamma}_1 \leq \bar{\gamma}_1 < 0$.

Suppose (12) holds. Let $\Gamma = \mathrm{diag}(\tilde{\Gamma}, 0, \ldots, 0)$, where $\tilde{\Gamma}$ is a 2×2 symmetric matrix whose elements are of the forms

$$\tilde{\gamma}_{11} = -\mu\beta z_1^{op}, \quad \tilde{\gamma}_{22} = -\mu\beta z_2^{op}, \quad \tilde{\gamma}_{12} = \sqrt{\tilde{\gamma}_{11}\tilde{\gamma}_{22}}. \quad (14)$$

where $0 \leq \mu \leq 1$ and β is equal to $2\underline{\gamma}_1$ if $0 < \underline{\gamma}_1 \leq \bar{\gamma}_1$, or equal to $2\bar{\gamma}_1$ if $\underline{\gamma}_1 \leq \bar{\gamma}_1 < 0$. Note that the eigenvalues of Γ are $\{0, -\mu\beta\}$. The zero eigenvalues obviously belong to Λ. To see $-\mu\beta$ also satisfies

$$|\lambda|^2 + 2\pi_1 \mathrm{Re}\,\lambda - 2\pi_2 \mathrm{Im}\,\lambda \leq 0$$

for all $\pi_i \in [\underline{\gamma}_i, \bar{\gamma}_i]$, $i = 1, 2$, note that $-\mu\beta$ is real and

$$(\mu\beta)^2 + 2\pi_1(-\mu\beta) = \begin{cases} 2\mu\underline{\gamma}_1(\underline{\gamma}_1 - \pi_1) & \text{if } 0 < \underline{\gamma}_1 \leq \bar{\gamma}_1 \\ 2\mu\bar{\gamma}_1(\bar{\gamma}_1 - \pi_1) & \text{if } \underline{\gamma}_1 \leq \bar{\gamma}_1 < 0 \end{cases}$$

Hence $(\mu\beta)^2 + 2\pi_1(-\mu\beta) \leq 0$ for all $\underline{\gamma}_1 \leq \pi_1 \leq \bar{\gamma}_1$, and we conclude that $\Gamma \in \mathcal{G}$ for all $\mu \in [0, 1]$. We will now show that $\det(I - \Gamma\Delta) = 0$ for some $\mu \in (0, 1]$.

Given the structure of matrix Γ, we observe that

$$\begin{aligned}&\det(I - \Gamma\Delta) = \det\left(I - \tilde{\Gamma} \cdot \mathrm{diag}(\Delta_1, \Delta_2)\right)\\ &= \det\left(\begin{bmatrix} 1 - \tilde{\gamma}_{11}\Delta_1 & -\tilde{\gamma}_{12}\Delta_2 \\ -\tilde{\gamma}_{12}\Delta_1 & 1 - \tilde{\gamma}_{22}\Delta_2 \end{bmatrix}\right) = 1 - \tilde{\gamma}_{11}\Delta_1 - \tilde{\gamma}_{22}\Delta_2\\ &= 1 + \mu\beta(z_1^{op}\Delta_1 + z_2^{op}\Delta_2)\end{aligned}$$

The second and the last equalities follow $\tilde{\gamma}_{12} = \sqrt{\tilde{\gamma}_{11}\tilde{\gamma}_{22}}$ and the expression of $\tilde{\gamma}_{11}$ and $\tilde{\gamma}_{22}$.

Since Δ_1 and Δ_2 satisfy (12), we see that

$$\begin{aligned}&\det(I - \Gamma\Delta) = 1 + \mu\beta(z_1^{op}\mathrm{Re}\,\Delta_1 + z_2^{op}\mathrm{Re}\,\Delta_2)\\ &= \begin{cases} 1 + \mu \cdot 2\underline{\gamma}_1(z_1^{op}\mathrm{Re}\,\Delta_1 + z_2^{op}\mathrm{Re}\,\Delta_2) & \text{if } 0 < \underline{\gamma}_1 \leq \bar{\gamma}_1 \\ 1 + \mu \cdot 2\bar{\gamma}_1(z_1^{op}\mathrm{Re}\,\Delta_1 + z_2^{op}\mathrm{Re}\,\Delta_2) & \text{if } \underline{\gamma}_1 \leq \bar{\gamma}_1 < 0 \end{cases}\end{aligned}$$

Therefore, the determinant of $I - \Gamma\Delta$ is equal to 1 when $\mu = 0$, and is less than or equal to 0 when $\mu = 1$. Hence, the determinate of $I - \Gamma\Delta$, as a function of μ, has a zero crossing in the interval $(0, 1]$, which in turn implies that $\det(I - \Gamma\Delta(j\omega_0))$ must take 0 value for a $\mu \in (0, 1]$. Since the network is at best neutrally stable at the zero crossing, we have constructed a connection matrix Γ which leads to a network that is not strictly stable. This concludes the proof for scenario (12).

For the case where (13) holds, the de-stabilizing interconnection matrix has the form $\mathrm{diag}(j \cdot \tilde{\Gamma}, 0, \ldots, 0)$, where elements of $\tilde{\Gamma}$ are of the form (14). Parameter μ is again a real number ranging between 0 and 1, whilst β is equal to $2\underline{\gamma}_2$ if $0 < \underline{\gamma}_2 \leq \bar{\gamma}_2$, or equal to $2\bar{\gamma}_2$ if $\underline{\gamma}_2 \leq \bar{\gamma}_2 < 0$. The arguments for showing that the interconnection is not strictly stable are completely analogous to those of scenario (12).

6 Concluding Remarks

Robust stability analysis of a class of interconnected systems is considered. It is assumed that all subsystems of such interconnected systems are linear-time-invariant, single-input-single-output, bounded, and causal operators on the space $\mathbf{L}_{2e}[0, \infty)$. Under the assumption that the network connection matrix is normal, a scalable robust stability criterion is proposed, where the complexity of verifying conditions for robust stability grows only *linearly* in the number of subsystems. This robust stability criterion can be characterized graphically as a polytope separation criterion on the point-wise convex hull of the Nyquist curves. Finally, we also show that such characterization is *necessary* in the sense that, if the stability criterion is violated, then there exists an interconnected configuration which leads to an unstable system.

References

[1] P. Barooah and J. Hespanha. Error amplification and disturbance propagation in vehicle strings with decentralized linear control. In *44th IEEE Conference on Decision and Control*, Seville, Spain, 2005.

[2] R. Cogill, S. Lall, and P. A. Parrilo. On structured semidefinite programs for the control of symmetric systems. In *Proceedings of the Allerton Conference on Communication, Control, and Computing*, pages 1536–1545, 2003.

[3] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.

[4] J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465 – 1476, 2004.

[5] A. Gattami and R. Murray. A frequency domain condition for stability of interconnected MIMO systems. In *Proceedings of the 2004 American Control Conference*, pages 3723–3727, Boston, MA, USA, 2004.

[6] Lestas I. and G. Vinnicombe. Scalability in heterogeneous vehicle platoons. In *American Control Conference*, July 2007.

[7] U. Jönsson, C.-Y. Kao, and H. Fujioka. Low dimensional stability criteria for large-scale interconnected systems. In *Proceedings of the European Control Conference*, Kos, Greece, 2007.

[8] I. Lestas and G. Vinnicombe. Scalable robustness for consensus protocols with heterogenous dynamics. In *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic, 2004.

[9] I Lestas and G. Vinnicombe. Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems. *IEEE transactions on automatic control*, 51(10):1613–1625, 2006.

[10] S. H. Low, F. Paganini, and J. C. Doyle. Internet congestion control. *Control Systems Magazine*, 22(1):28–43, 2002.

[11] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control*, 42(6):819–830, June 1997.

[12] A. Pogromsky and H. Nijmeijer. Cooperative oscillatory behavior of mutually coupled dynamical systems. *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, 48(2):152–162, 2001.

[13] R. Olfati Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions On Automatic Control*, 2004.

[14] P. Seiler, A. Pant, and K. Hedrick. Disturbance propagation in vehicle strings. *IEEE Transactions on Automatic Control*, 49(10):1835–1842, 2004.

[15] Y.-P. Tian and H.-Y. Yang. Stability of the internet congestion control with diverse delays. *Automatica*, 40(9):1533–1541, 2004.

[16] G. Vinnicombe. On the stability of end-to-end congestion control for the internet. Technical Report CUED/F-INFENG/TR.398, University of Cambridge, 2000.

[17] C. W. Wu and L. O. Chua. Synchronization in an array of linearly coupled dynamical systems. *IEEE Transactions on Circuits and Systems-I: Fundamental Theor y and Applications*, 42(8):430–447, 1995.