IQC CHARACTERIZATIONS OF SIGNAL CLASSES

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Abstract

Worst case average performance analysis is considered in this paper. The disturbance in the system is assumed to belong to a "class of inputs signals". A class of signals is here defined to be a set of families of signals, where each family satisfies an average spectral constraint. Exact characterizations in terms of integral quadratic constraints (IQC) are given for a class of white signals and a class of signals that are generated by autonomous linear systems. The IQCs are defined in terms of multipliers and important issues in numerical optimization of the multipliers are discussed in the paper.

1 Introduction

In order to reduce conservatism in robust performance analysis we need to exploit information on the disturbance signals. This is not always straightforward. For example, if the disturbance is of white noise type then it is natural to use a stochastic signal model for the performance criterion while it is more convenient to consider deterministic signals for the stability robustness. This problem has received much attention in connection to the robust $\mathbf{H}_2$ performance problem, see, for example, [Meg92, ZGBD94, Fer97, Pag99, Pag96b], and [Pag96a].

We consider performance analysis in a standard framework of Integral Quadratic Constraints (IQC), see [MR97]. This means that we work with square integrable functions. Many signals of practical importance are not directly representable as such functions. We introduce the concept of "signal classes" to cope with this problem. A class of signals is defined to be a set of families of signals, where the signals in each family satisfies some average spectral property. In this way we can define the class of white signals to be such that each family has an average spectrum which is flat over some bandwidth and zero outside this bandwidth. The performance analysis must now be considered as the worst case average over the class of signals.

We show that the class of white signals and a class of exponentially decaying signals are uniquely defined in terms of IQCs. The latter signal class can, for example, be used to account for uncertain initial conditions in the linear part of the system.

We will discuss how our result on white signals are related to robust $\mathbf{H}_2$ analysis. In fact, the resulting robustness condition is similar to Paganinis in [Pag99]. Paganini consider sets of almost white signals and then let the accuracy tend to zero. Sufficient, and for some special cases also necessary, conditions for robust $\mathbf{H}_2$ performance is then obtained in [Pag99] by letting the bandwidth tend to infinity. Making the bandwidth tend to infinity thus gives strong results but it also makes verification of the robustness constraint harder.

Our analysis results can be applied to a large class of nonlinear and uncertain systems. The performance conditions are easily implemented in terms of convex optimization problems. Suitable parametrizations of the multipliers that define the IQCs are discussed in detail.

Notation

We let $\mathbf{L}_2^m[0,\infty)$ denote the vector space of square integrable $\mathbf{R}^m$ valued functions. The norm on $\mathbf{L}_2^m[0,\infty)$ is defined as

$$\|f\| = \int_0^\infty f(t)^T f(t) dt.$$

The bi-infinite space $\mathbf{L}_2^m(-\infty,\infty)$ is defined similarly. We will use the notation $\mathbf{L}_2$ to mean either of these two vector spaces. A causal operator $\Delta : \mathbf{L}_2 \to \mathbf{L}_2^m[0,\infty)$ is bounded if there exists $c > 0$ such that $\|\Delta(v)\| \leq c\|v\|$, for all $v \in \mathbf{L}_2^m$. The following standard notation is used:

$\mathbf{RL}_\infty^{m\times m}$ The vector space of proper real rational transfer functions with no poles on the imaginary axis.

$\mathbf{RH}_\infty^{m\times m}$ The subspace of $\mathbf{RL}_\infty^{m\times m}$ consisting of functions with no poles in the closed right half plane.

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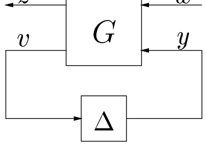


Figure 1: System for performance analysis.

$\mathbf{RH}_2^{m\times m}$ Consists of the strictly proper transfer functions in $\mathbf{RH}_\infty^{m\times m}$.

$\mathbf{S}_\infty^{m\times m}$ The subset $\{H \in \mathbf{RL}_\infty^{m\times m} : H = H^*\}$, where the adjoint is defined as $H^*(s) = H(-s)^T$.

The $\mathbf{H}_2$-norm of $H \in \mathbf{RH}_2$ is defined by $\|H\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{tr}(G(j\omega)^* G(j\omega))d\omega$, where $\mathrm{tr}(\cdot)$ is the trace of a matrix. We define the Fourier transform of $w \in \mathbf{L}_2^n$ as

$$\widehat{w}(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t} w(t)dt.$$

2 Average Performance Analysis

We will here discuss worst case average performance analysis using IQCs. Consider the system, see also Figure 1,

$$\begin{aligned} \begin{bmatrix} z \\ v \end{bmatrix} &= G\begin{bmatrix} w \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}\begin{bmatrix} w \\ y \end{bmatrix}, \\ y &= \Delta(v), \end{aligned} \tag{1}$$

where it is assumed that Δ is bounded and causal on $\mathbf{L}_2^m$ and G is a bounded and causal operator with transfer function in $\mathbf{RH}_\infty^{(n+m)\times(n+m)}$. We assume for simplicity that the feedback loop is stable, i.e., the map from w to z is bounded on $\mathbf{L}_2^n$. We want to compute the worst case average performance of this system over a class of input signals. A class of signals is defined as follows.

Definition 1. A signal class is a set of families of signals $\mathcal{W} = \{w : w = \{w_1, \ldots, w_N\}, w_i \in \mathbf{L}_2^n\}$, where each family satisfies an average spectral constraint

$$\frac{1}{N}\sum_{i=1}^{N} \widehat{w}_i(j\omega)\widehat{w}_i(j\omega)^* \subset \mathcal{S}$$

where $\mathcal{S} \subset \mathbf{S}_\infty^{n\times n}$.

Let σ_p be a time-invariant quadratic form that defines the performance constraint. For example, if

$$\sigma_p(z, w) = \|z\|^2 - \gamma^2\|w\|^2,$$

then the average IQC constraint

$$\frac{1}{N}\sum_{i=1}^{N} \sigma_p(z_i, w_i) \le 0 \tag{2}$$

for all $w = \{w_1, \ldots, w_N\} \in \mathcal{W}$ and corresponding outputs z_i, implies that the worst case average $\mathbf{L}_2$-gain of system (1) over the class $\mathcal{W}$ is γ.

We will use IQCs to derive sufficient conditions for robust average performance. The perturbation Δ is said to satisfy the IQC defined by σ_Δ ($\Delta \in \mathrm{IQC}(\sigma_\Delta)$) if

$$\sigma_\Delta(v, \Delta(v) \ge 0,\ v \in \mathbf{L}_2^m[0,\infty).$$

We will consider quadratic forms on the form

$$\sigma_\Delta(v, \Delta(v)) = \int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v} \\ \widehat{\Delta(v)} \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \widehat{v} \\ \widehat{\Delta(v)} \end{bmatrix} d\omega, \tag{3}$$

where $\Pi \in \mathbf{S}_\infty^{2m\times 2m}$.

Similarly, the signal class $\mathcal{W}$ is said to satisfy the IQC defined by $\sigma_\mathcal{W}$ ($\mathcal{W} \in \mathrm{IQC}(\sigma_\mathcal{W})$) if

$$\frac{1}{N}\sum_{i=1}^{N} \sigma_\mathcal{W}(w_i) \ge 0 \tag{4}$$

for all $w = \{w_1, \ldots, w_N\} \in \mathcal{W}$. We consider IQCs on the form

$$\sigma_\mathcal{W}(w) = \int_{-\infty}^{\infty} \widehat{w}^* \Psi(j\omega)\widehat{w} d\omega, \tag{5}$$

where $\Psi \in \mathbf{S}_\infty^{n\times n}$. The next proposition gives a sufficient condition for robust average performance.

Proposition 1. *Assume that the system in (1) is stable and that the input w belongs to the signal class $\mathcal{W}$. If $\Delta \in IQC(\sigma_\Delta)$ and $\mathcal{W} \in IQC(\sigma_\mathcal{W})$ then the performance criterion in (2) is satisfied if*

$$\sigma(Gu, u) = \sigma_p(G_1 u, w) + \sigma_\Delta(G_2 u, y) + \sigma_\mathcal{W}(w) \le 0, \tag{6}$$

for all $u = (w^T, y^T)^T \in \mathbf{L}_2^{n+m}$. Here $G_1 = \begin{bmatrix} G_{11} & G_{12} \end{bmatrix}$ and $G_2 = \begin{bmatrix} G_{21} & G_{22} \end{bmatrix}$.

Proof. Assume $w = \{w_1, \ldots, w_N\} \in \mathcal{W}$ and let the resulting signals in the closed loop system be denoted z_i, v_i, y_i, and finally let $u_i = (w_i^T, y_i^T)$. Since $G_2 u_i = v_i$ and $y_i = \Delta(v_i)$ we have

$$\frac{1}{N}\sum_{i=1}^{N} \sigma_\Delta(G_2 u_i, y_i)) = \frac{1}{N}\sum_{i=1}^{N} \sigma_\Delta(v_i, \Delta(v_i)) \ge 0,$$

and since $w \in \mathcal{W}$

$$\frac{1}{N}\sum_{i=1} \sigma_\mathcal{W}(w_i) \ge 0.$$

Hence, (6) implies that (2) is satisfied since $\sigma_p(G_1 u_i, w_i) = \sigma_p(z_i, w_i)$. This proves the proposition. ☐

Remark 1. Solving (6) with strict inequality (i.e., $\sigma(Gw, w) \le -\varepsilon\|w\|^2$, for some $\varepsilon > 0$) is often advantageous from a computational point of view. Another advantage with using strict inequality is that it ensures both robust performance and robust stability of the system (1) under some weak conditions.

Remark 2. If σ_Δ is defined by Π and $\sigma_{\mathcal{W}}$ is defined by Ψ then (6) can be formulated as

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11} & 0 & \Pi_{12} \\ \hline 0 & 0 & -\gamma^2 I + \Psi & 0 \\ 0 & \Pi_{12}^* & 0 & \Pi_{22} \end{array}\right] \begin{bmatrix} G \\ I \end{bmatrix} (j\omega) \leq 0, \quad (7)$$

for all $\omega \in [0, \infty]$.

Remark 3. In general, we have many IQCs and the robustness conditions in (6) (or equivalently (7)) should be stated as a convex feasibility problems over the set of IQCs. To do this we assume that we have finite parametrizations $\sum_{i=1}^{N_1} \lambda_i \Psi_i$ and $\sum_{i=1}^{N_2} \kappa_i \Pi_i$, where the parameter vectors satisfy $\lambda \in \Lambda$, and $\kappa \in K$, for appropriate convex cones Λ and K. The robustness condition can be formulated as the feasibility test: Find $\lambda \in \Lambda$ and $\kappa \in K$ such that

$$\sigma_p(G_1 w, w_1) + \sum_{i=1}^{N_1} \kappa_i \sigma_\Delta^i(G_2 w, w_2) + \sum_{i=1}^{N_2} \lambda_i \sigma_{\mathcal{W}}^i(w_1) \leq 0,$$

where σ_Δ^i is defined as in (3) with Π replaced by Π_i and where $\sigma_{\mathcal{W}}^i$ is defined as in in (5) with Ψ replaced by Ψ_i. This feasibility test can be formulated as a parameter dependent LMI on the same form as in (7). A software package for solving the strict version of this LMI is available, see [MKJR97].

3 A Class of White Signals

We will here discuss how deterministic white noise signals can be represented as $\mathbf{L}_2$-signals with flat spectrum over a suitably large frequency range. For the scalar case we say that a signal $w \in \mathbf{L}_2(-\infty, \infty)$ is white over the frequency range $-b \leq \omega \leq b$ if

$$|\widehat{w}(j\omega)|^2 = \begin{cases} \frac{\pi}{b}\|w\|^2, & \omega \in [-b, b] \\ 0, & |\omega| > b \end{cases} \quad (8)$$

The situation is less trivial in higher dimensions. We would like to say that $w \in \mathbf{L}_2^n(-\infty, \infty)$ is white over the frequency range $-b \leq \omega \leq b$ if

$$\widehat{w}(j\omega)\widehat{w}(j\omega)^* = \begin{cases} \frac{\pi}{bn}\|w\|^2 I, & \omega \in [-b, b] \\ 0, & |\omega| > b \end{cases}.$$

This is, however, impossible since the left hand side is a rank one matrix. To overcome this problem we define the following class of white signals.

Definition 2. In the class $\mathcal{W}_{\text{white}} = \{w : w = \{w_1, \ldots, w_N\}, w_i \in \mathbf{L}_2^n(-\infty, \infty)\}$, each family satisfies

$$\frac{1}{N}\sum_{i=1}^{N} \widehat{w}_i(j\omega)\widehat{w}_i(j\omega)^* = \begin{cases} cI, & |\omega| \leq b, \\ 0, & |\omega| > b. \end{cases} \quad (9)$$

where $c = \pi/(bnN)\sum_{i=1}^{N} \|w_i\|^2$.

This means that each family in the class has an average spectrum which is flat over the bandwidth $[-b, b]$. That is, the energy is concentrated to the interval $[-b, b]$ with equal distribution between the components of the signals and there is no *cross correlation* between the components.

Note that it is possible to find n signals $w_1, \ldots, w_n$ such that the average satisfies the property (9). We just need to let $w_i = w e_i$, where e_i is the i^{th} unit vector and w is a scalar signal satisfying (8).

The next proposition gives an exact characterization of the class $\mathcal{W}_{\text{white}}$.

Proposition 2. *The IQCs defined by the multipliers*

$$\Psi_{\text{white}} = \left\{ \Psi \in \boldsymbol{S}_\infty^{n\times n} : \int_{-b}^{b} tr(\Psi(j\omega))d\omega \geq 0 \right\}. \quad (10)$$

give an exact characterization of $\mathcal{W}_{\text{white}}$.

Proof. We need to prove two things

(*i*) For every $\Psi \in \Psi_{\text{white}}$ we have

$$\frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} \widehat{w}_i^* \Psi \widehat{w}_i d\omega \geq 0, \quad (11)$$

for any family $w = \{w_1, \ldots, w_N\} \in \mathcal{W}_{\text{white}}$.

(*ii*) If $w_1, \ldots, w_N \in \mathbf{L}_2^n(-\infty, \infty)$ violates (9), then there exists $\Psi \in \Psi_{\text{white}}$ such that

$$\frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} \widehat{w}_i^* \Psi \widehat{w}_i d\omega < 0, \quad (12)$$

To prove (*i*) we just note that

$$\frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} \widehat{w}_i^* \Psi \widehat{w}_i d\omega = \int_{-\infty}^{\infty} \text{tr}\left(\Psi(\frac{1}{N}\sum_{i=1}^{N}\widehat{w}_i\widehat{w}_i^*)\right) d\omega$$
$$= \frac{\pi}{bn}(\frac{1}{N}\sum_{i=1}^{N}\|w_i\|^2)\int_{-b}^{b}\text{tr}(\Psi)d\omega \geq 0.$$

For the proof of (*ii*) we define the function

$$Z_0(\omega) = \begin{cases} bI, & \omega > b, \\ \omega I, & |\omega| \leq b, \\ -bI, & \omega < -b. \end{cases} \quad (13)$$

This is a function of bounded variation and it belongs to the dual space of $\mathbf{S}_\infty^{n\times n}$. It defines a linear functional on $\mathbf{S}_\infty^{n\times n}$ in terms of the Stieltjes integral

$$\langle \Psi, Z_0 \rangle = \int_{-\infty}^{\infty} \text{tr}(\Psi(j\omega)dZ_0(\omega)).$$

We note that

$$\langle \Psi, Z_0 \rangle = \int_{-b}^{b} \text{tr}(\Psi(j\omega))d\omega \geq 0,$$

which shows that Ψ_{white} is the half space

$$\Psi_{\text{white}} = \left\{ \Psi \in \mathbf{S}_{\infty}^{n\times n} : \langle \Psi, Z_0 \rangle \geq 0 \right\}.$$

Now let $w_1, \ldots, w_N \in \mathbf{L}_2^n(-\infty, \infty)$ be a set of signals that does not satisfy (9) and define

$$Z(\omega) = \frac{1}{N} \sum_{i=1}^{N} \int_0^{\omega} \widehat{w}_i \widehat{w}_i^* d\nu \tag{14}$$

This is a function of bounded variation that is different from Z_0, and it thus defines another half space. It follows that there exists $\Psi \in \Psi_{\text{white}}$ such that $\langle \Psi, Z \rangle < 0$. $\square$

4 Robust $\mathbf{H}_2$-performance

Average analysis with our class of white signals gives a robustness condition that is analogous to Paganinis result in [Pag99]. To see this and the corresponding connection to robust $\mathbf{H}_2$ analysis we consider the case when the uncertainty $\Delta \in \mathbf{RH}_{\infty}^{m\times m}$ satisfies the quadratic constraint

$$\begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I \\ \Delta(j\omega) \end{bmatrix} \geq 0, \ \forall \omega.$$

If we let $Y = \gamma^2/(n2b)I - \Psi$ then we have

$$\int_{-b}^{b} \operatorname{tr}(Y(j\omega)) d\omega \leq \gamma^2,$$

and our condition for the average $\mathbf{L}_2$-gain to be less than $\gamma/\sqrt{2nb}$ becomes

$$\begin{bmatrix} G(j\omega) \\ I \end{bmatrix}^* \left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & \Pi_{11} & 0 & \Pi_{12} \\ \hline 0 & 0 & -Y(j\omega) & 0 \\ 0 & \Pi_{12}^* & 0 & \Pi_{22} \end{array}\right] \begin{bmatrix} G(j\omega) \\ I \end{bmatrix} \leq 0, \ \forall \omega. \tag{15}$$

Let $Q = \Delta(I - G_{22}\Delta)^{-1}G_{21}$ and multiply the above inequality with $V^* = \begin{bmatrix} I & Q^* \end{bmatrix}$ on the left, V on the right and then integrate the trace from $-b$ to b. This gives

$$\int_{-b}^{b} \operatorname{tr}(G_{\Delta}(j\omega)^* G_{\Delta}(j\omega)) d\omega \leq \int_{-b}^{b} \operatorname{tr}(Y(j\omega)) d\omega \leq \gamma^2,$$

where G_{Δ} is the linear fractional transformation that represents the closed loop system

$$G_{\Delta} = G_{11} + G_{12}\Delta(I - G_{22}\Delta)^{-1}G_{21}.$$

Hence, as $b \to \infty$ we get $\|G_{\Delta}\|_{\mathbf{H}_2} \leq \gamma$. The criterion in (15) is thus closely related to the one obtained in in [Pag99], where sufficient and for some cases even necessary conditions for robust $\mathbf{H}_2$ performance are derived. The price paid to obtain such strong conclusions is that Y needs to integrable over the imaginary axis with $\int_{-\infty}^{\infty} \operatorname{tr}(Y(j\omega)) d\omega \leq \gamma^2$. This will in general complicate the verification of (15) since Y will be strictly proper. Average analysis with signals that are white only over a finite bandwidth is generally much simpler.

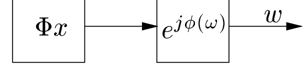


Figure 2: Generation of output signals from an autonomous system. We assume that $\Phi \in \mathbf{RH}_2^{m\times n}$, $x \in \mathbf{R}^n$, and that $\phi : \mathbf{R} \to \mathbf{R}$ is odd and measurable. This means that $e^{j\phi(\omega)}$ gives a frequency varying phase adjustment of the signal. It is assumed that ϕ is chosen such that $w \in \mathbf{L}_2^m[0, \infty)$. We assume that x and ϕ can vary from experiment to experiment.

5 IQCs for Autonomous Systems

We will in this section derive an exact characterization of a class of signals that can be viewed as outputs of the autonomous systems in Figure 2. By doing average analysis along the lines of Proposition 1, we get an estimate of what can be expected when the autonomous system generates inputs to the system in (1).

Definition 3. Assume $\Phi \in \mathbf{RH}_2^{m\times n}$ and define the class $\mathcal{W}_{\Phi} = \{w : w = \{w_1, \ldots, w_N\},\ w_i \in \mathbf{L}_2^m[0, \infty)\}$, where any family $w = \{w_1, \ldots, w_N\}$ satisfies the following average spectral condition

$$\frac{1}{N} \sum_{i=1}^{N} \widehat{w}_i \widehat{w}_i^* \in \overline{\operatorname{co}}(S_{\Phi}). \tag{16}$$

Here $\overline{\operatorname{co}}$ denotes the closed convex hull and $S_{\Phi} = \{\Phi x x^T \Phi^* : x \in \mathbf{R}^n\}$.

These signals are exponentially decaying in the following sense

Lemma 1. *There is a positive constant α such that $e^{\alpha t} w_i \in \boldsymbol{L}_2^m[0, \infty)$, for every member of the families $w = \{w_1, \ldots, w_N\} \in \mathcal{W}_{\Phi}$.*

Proof. It follows from Definition 3 that we can assume

$$\frac{1}{N} \sum_{i=1}^{N} \widehat{w}_i \widehat{w}_i^* = \sum_{i=1}^{n} \Phi x_i x_i^T \Phi^*$$

for some $x_i \in \mathbf{R}^n$. Let α be such that $\Phi(s - \alpha) \in \mathbf{RH}_2^{m\times n}$. Then $w_{\alpha i} = e^{\alpha t} w_i \in \mathbf{L}_2^m[0, \infty)$, $i = 1, \ldots, N$, since

$$\begin{aligned} \frac{2\pi}{N} \sum_{i=1}^{N} \|w_{\alpha i}\|^2 &= \int_{-\infty}^{\infty} \frac{1}{N} \sum_{i=1}^{N} \operatorname{tr}(\widehat{w}_{\alpha i} \widehat{w}_{\alpha i}^*) d\omega) \\ &= \sum_{i=1}^{n} x_i^T \int_{-\infty}^{\infty} \Phi^*(j\omega - \alpha)\Phi(j\omega - \alpha) d\omega x_i < \infty. \end{aligned}$$

This proves the lemma. $\square$

Remark 4. Note that it is impossible to prove exponential decay of the form $|w(t)| \leq ce^{-\alpha t}$ for some positive constants c and α. The reason is that we allow an arbitrary time delay in the signal. To see this, consider signals with Fourier transforms $\widehat{w}_1$ and $\widehat{w}_2 = e^{-j\omega T}\widehat{w}_1(j\omega)$. We have $\widehat{w}_1\widehat{w}_1^* = \widehat{w}_2\widehat{w}_2^*$, and the conclusion follows since $w_2(t) = w_1(t-T)$, and T can be arbitrarily large.

The IQCs for the signals in Definition 3 should be defined in the the average sense (4). We have the following unicity result

Proposition 3. *Let $\Phi \in \boldsymbol{RH}_2^{m\times n}$ be given. The signals in $\mathcal{W}_\Phi$ are uniquely defined in terms of the multipliers*

$$\Psi_\Phi = \left\{ \Psi \in \boldsymbol{S}_\infty^{m\times m} : \int_{-\infty}^{\infty} \Phi^*\Psi\Phi d\omega \geq 0 \right\}.$$

Proof. We will first give an alternative characterization of the multipliers in Ψ_Φ. We notice that $\Psi \in \Psi_\Phi$ if and only if

$$\begin{aligned} &x^T\left(\int_{-\infty}^{\infty}\Phi^*\Psi\Phi d\omega\right)x \\ &= \int_{-\infty}^{\infty} \mathrm{tr}\left(\Psi(\Phi x x^T\Phi^*)\right)d\omega = \langle \Psi, Z\rangle \geq 0,\ \forall x \in \mathbf{R}^n, \end{aligned} \tag{17}$$

where $Z(\omega) = \int_0^\omega \Phi(i\nu)xx^T\Phi(i\nu)^* d\nu$ belongs to the dual space of $\mathbf{S}_\infty^{n\times n}$ and the linear functional $\langle\cdot,\cdot\rangle$ is defined in terms of the Stieltjes integral

$$\langle \Psi, Z\rangle = \int_{-\infty}^{\infty} \mathrm{tr}(\Psi(j\omega)dZ(\omega)).$$

It follows from (17) that

$$\begin{aligned} \Psi_\Phi &= \cap_{Z\in Z_\Phi}\left\{\Psi \in \mathbf{S}_\infty^{m\times m} : \langle\Psi, Z\rangle \geq 0\right\} \\ &= \cap_{Z\in\overline{\mathrm{co}}(Z_\Phi)}\left\{\Psi \in \mathbf{S}_\infty^{m\times m} : \langle\Psi, Z\rangle \geq 0\right\}, \end{aligned} \tag{18}$$

where Z_Φ is defined as

$$Z_\Phi = \left\{ Z(\omega) = \int_0^\omega \Phi x x^T \Phi^* d\nu : x \in \mathbf{R}^n \right\}.$$

We are now ready to prove the claim of the proposition. Assume that $w = \{w_1, \ldots, w_N\} \in \mathcal{W}_\Phi$ and $\Psi \in \Psi_\Phi$. Then

$$\frac{1}{N}\sum_{i=1}^N \int_{-\infty}^{\infty} \widehat{w}_i^*\Psi\widehat{w}_i d\omega = \langle \Psi, Z_w\rangle, \tag{19}$$

where

$$Z_w(\omega) = \frac{1}{N}\sum_{i=1}^N \int_0^\omega \widehat{w}_i\widehat{w}_i^* d\nu. \tag{20}$$

It follows by the definition of $\mathcal{W}_\Phi$ that $Z_w \in \overline{\mathrm{co}}(Z_\Phi)$. Hence, it follows from (18) that (19) is positive.

For the other direction assume that $w_1, \ldots, w_N$ does not satisfy (16). If we define Z_w as in (20) with these w_i then $Z_w \notin \overline{\mathrm{co}}(Z_\Phi)$. We will show that there exists $\Psi \in \Psi_\Phi$ such that $\langle\Psi, Z_w\rangle < 0$.

To do this we first note that[1]

$$\cap_{\mathrm{co}(Z_\Phi\cup Z_w)}\{\Psi : \langle\Psi, Z\rangle \geq 0\} \subset \Psi_\Phi$$

where the inclusion is proper. The properness of the inclusion follows since the convex cone Ψ_Φ is the intersection of the half spaces $H_Z = \{\Psi \in \mathbf{S}_\infty^{m\times m} : \langle\Psi, Z\rangle \geq 0\}$, $Z \in \overline{\mathrm{co}}(Z_\Phi)$, and the first convex cone is obtained by intersecting Ψ_Φ with the halfspace $H_w = \{\Psi \in \mathbf{S}_\infty^{m\times m} : \langle\Psi, Z_w\rangle \geq 0\}$, which is different from the H_Z's since by assumption $Z_w \notin \overline{\mathrm{co}}(Z_\Phi)$. Hence, there exists nonzero

$$\widetilde{\Psi} \in \Psi_\Phi \setminus \cap_{\mathrm{co}(Z_\Phi\cup Z_w)}\{\Psi : \langle\Psi, Z\rangle \geq 0\}.$$

We obviously have $\left\langle \widetilde{\Psi}, Z_w\right\rangle < 0$. □

6 Numerical Issues

We want to parametrize a finite-dimensional subset of Ψ_{white}. One such parametrization is $\Psi = Y + Y^*$, where

$$Y(s) = X_0 + \sum_{i=1}^N \frac{1}{2}\left(\frac{Z_i}{s+a_i} + \frac{\overline{Z}_i}{s+\overline{a}_i}\right)$$

where the a_i are distinct with Re $a_i > 0$ and $Z_i = X_i + iY_i$, $X_i, Y_i \in \mathbf{R}^{n\times n}$. We note that

$$\begin{aligned} &\frac{1}{2}\left(\frac{Z_i}{s+a_i} + \frac{\overline{Z}_i}{s+\overline{a}_i}\right) \\ &\quad = \begin{cases} \frac{X_i}{s+a_i}, & a_i \in \mathbf{R}, \\ \dfrac{sX_i + \mathrm{Re}(a_i)X_i + \mathrm{Im}(a_i)Y_i}{s^2 + 2\mathrm{Re}(a_i)s + |a_i|^2}, & \text{otherwise.} \end{cases} \end{aligned}$$

This means that $Y \in \mathbf{RH}_\infty^{n\times n}$ and in fact, this is the most general way to construct an $n\times n$ transfer function with distinct poles.

The values of Z_i must be constrained such that $\int_{-b}^b \mathrm{tr}(\Psi)d\omega \geq 0$. We can obtain an efficient characterization for this constraint. Let us first consider the case when $a_i \in \mathbf{R}$. We have

$$\int_{-b}^b \frac{1}{j\omega + a_i}d\omega = 2\int_0^b \frac{a_i}{\omega^2 + a_i^2}d\omega = 2\mathrm{arctan}(b/a_i). \tag{21}$$

The left hand side of (21) is an analytic function of a_i in the region Re $a_i > 0$. It follows by analytic continuation that (21) holds for all Re $a_i > 0$. Hence,

$$\begin{aligned} &\int_{-b}^b \mathrm{tr}(Y(j\omega) + Y(j\omega)^*)d\omega = 2\int_{-b}^b \mathrm{tr}(Y(j\omega))d\omega \\ &\qquad = 4b\mathrm{tr}(X_0) + 4\sum_{i=1}^N \mathrm{Re}(\mathrm{tr}(Z_i)\cdot\mathrm{arctan}(b/a_i)). \end{aligned}$$

[1] Note that both sets are nontrivial convex cones. To see this let $P = \{\Psi \in \mathbf{S}_\infty^{m\times m} : \Psi(j\omega) \geq 0,\ \forall\omega\}$. Then it is easy to verify that $P \subset \cap_{\mathrm{co}(Z_\Phi\cup Z_w)}\{\Psi : \langle\Psi, Z\rangle \geq 0\} \subset \cap_{\mathrm{co}(Z_\Phi)}\{\Psi : \langle\Psi, Z\rangle\} \geq 0\}$

Hence, the matrices X_i, Y_i must satisfy

$$\operatorname{tr}(bX_0 + \sum_{i=1}^{N} (\operatorname{Re}(\arctan(b/a_i))X_i - \operatorname{Im}(\arctan(b/a_i)Y_i))) \geq 0$$

We will next discuss how we can find suitable finite dimensional parametrizations of multipliers from the set Ψ_Φ. It is of particular interest to find a computationally inexpensive method to impose the constraint $\int_{-\infty}^{\infty} \Phi^* \Psi \Phi d\omega \geq 0$. This constraint can for general finite dimensional parametrizations be transformed into an equivalent constraint that involves a Lyapunov equation (or Lyapunov inequality) and an LMI constraint on the Lyapunov matrix, see for example [Jön96]. The number of decision variables will, however, grow fast with the number of states of $\Phi^* \Psi \Phi$. We will here show how this can be overcome by precomputing the Lyapunov equations for a basis of the multipliers.

We will only discuss the case when Ψ has a finite number of distinct stable real poles. Complex poles can be treated in exactly the same way. Let

$$\Psi = \sum_{k=1}^{N} \frac{X}{s + a_k} + \frac{X^T}{-s + a_k} = \sum_{k=1}^{N} \Psi_k^* M_k \Psi_k,$$

where $a_k > 0$,

$$\Psi_k(s) = \begin{bmatrix} I \\ \dfrac{1}{s + a_k} I \end{bmatrix}, \quad M_k = \begin{bmatrix} 0 & X_k \\ X_k^T & 0 \end{bmatrix},$$

and $X_k \in \mathbf{R}^{m \times m}$ is the variable. We can represent the M_k as linear combinations

$$M_k = \sum_{q=1}^{m} \sum_{r=1}^{m} x_{k_{qr}} \begin{bmatrix} 0 & E_{qr} \\ E_{qr}^T & 0 \end{bmatrix},$$

where $E_{qr} = e_q e_r^T$, i.e., all elements are zero except for the 1 at the qr^{th} position. We can now represent the vectors in our finite dimensional subset of Ψ_Φ as

$$\Psi = \sum_{k=1}^{N} \sum_{q,r} x_{k_{qr}} \Psi_k^* M_{qr} \Psi_k,$$

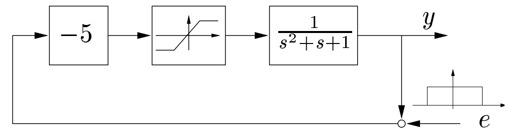
where $x_{k_{qr}} \in \mathbf{R}$ are the decision variables. The constraints $\int_{-\infty}^{\infty} \Phi^* \Psi \Phi d\omega \geq 0$ can now be represented as

$$\sum_{k=1}^{N} \sum_{qr} x_{k_{qr}} \widetilde{M}_{k_{qr}} \geq 0,$$

where $\widetilde{M}_{k_{qr}} = \int_{-\infty}^{\infty} \Phi^* \Psi_k^* M_{q_{qr}} \Psi_k \Phi d\omega$ can be precomputed as follows. Let $\Psi_k(s)\Phi(s) = C_k(sI - A_k)^{-1} B_k$, where A_k is stable since Ψ_k and Φ are assumed to be stable. Then $\widetilde{M}_{k_{qr}} = B_k^T P_{k_{qr}} B_k$, where

$$A_k^T P_{k_{qr}} + P_{k_{qr}} A_k + C_k^T M_{k_{qr}} C_k = 0.$$

7 Example




```
s=tf([1 0],1)
G=1/(s*s+s+1)
K=-5
b=50
abst_init_iqc;
e=iqc_white(1,b,[0.5+2.4*i]);
w=signal;
v=K*(e+G*w);
w==iqc_slope(v,3,1,0,1);
g=iqc_gain_tbx(e,w);
```

Poles	Energy gain
–	11.7665
$0.5 \pm 2.4i$	6.9243
several	6.5

8 Concluding Remarks

The suggested approach of average performance analysis over a class of input signals is a combination of deterministic and stochastic ideas. In fact, Definition 1 means that each member of the family $w = \{w_1, \ldots, w_N\}$ is equally likely.

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