Local Robustness of Hyperbolic Limit Cycles

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Summary. Local robustness of limit cycles are investigated for systems that can be modeled as a feedback interconnection of an exponentially stable linear system with a nonlinear function. Conditions are given under which the limit cycle and the number of unstable modes persist for sufficiently small dynamic perturbations.

1 Introduction

Stability and robustness of limit cycle oscillations are properties of fundamental importance in many applications in electronics, mechanics, biology, and physics. Limit cycle oscillation is crucial in control applications such as biological locomotion [6], rhythmic mechanical motion [7] and auto-tuning [1]. Tools for rigorous analysis of stability and robustness of limit cycle oscillations is important in the design and verification of such systems. The classical literature provides several useful results but few, if any of them, extends directly to system descriptions that are subject to various forms of unmodelled dynamics. One problem is that the classic results were derived in a state space formalism which does not extend easily to systems with unknown possible infinite dimension. Another problem is that the introduction of uncertainty in the system dynamics perturbs both the period time and the orbit of the limit cycle which is in stark contrast to the traditional problems in robust control where the equilibrium solution remains fixed when the system is perturbed. This makes robust stability analysis of limit cycles a challenging problem.

In this paper we briefly review and extend some of the results in [3]. There we proved that the well-known condition on the characteristic multipliers for robustness of finite dimensional systems extends to a class of systems with dynamic uncertainties. We also showed how bounds on a robustness margin can be estimated. Here we extend the local result to hold also in the case when the limit cycle is hyperbolic. In particular, we derive conditions under which the limit cycle and the number of unstable modes persist for sufficiently small dynamic perturbations.

Notation

We will let $C(1)$ denote the set of continuous one periodic functions equipped with the norm $\|v\|_{C(1)} = \sup_{t\in[0,1]} |v(t)|$. The exponentially weighted $\mathbf{L}_2$ space $\mathbf{L}_{2\alpha}[0,\infty) = \{e(t) : \int_0^\infty e^{2\alpha t}|e(t)|^2 dt < \infty\}$ will be used to define and prove exponential stability. The norm on the usual $\mathbf{L}_2[0,\infty)$ space is denoted $\|\cdot\|$ while the norm on $\mathbf{L}_{2\alpha}[0,\infty)$ is denoted and defined as $\|v\|_\alpha = (\int_0^\infty e^{2\alpha t}|v(t)|^2 dt)^{1/2}$. The spatial norm will always be the Euclidean norm $|v| = (\sum_{i=1}^n v_i^2)^{1/2}$. At several places we consider the space $C(1)\times\mathbf{R}$ with the norm $\|(v,T)\|_{C(1)\times\mathbf{R}} = (\|v\|_{C(1)}^2 + |T|^2)^{1/2}$. If X denotes a normed vector space then its dual X^* is the Banach space of all bounded linear functionals on X. If $g \in X^*$ and $x \in X$, then we use the notation $g(x) = \langle x, g\rangle$ for the functional.

We use that the *characteristic multipliers* of a periodic matrix $A(t) = A(t+T_0)$ are the eigenvalues of the monodromy matrix $\Phi(T_0,0)$, where

$$\frac{d}{dt}\Phi(t,0) = A(t)\Phi(t,0), \qquad \Phi(0,0) = I.$$

2 Model Assumptions

We consider systems consisting of a feedback interconnection of an exponentially stable linear time-invariant (LTI) plant and a memoryless nonlinearity

$$y(t) = \int_{-\infty}^{t} h(t-\tau,\theta)\varphi(y(\tau))\mathrm{d}\tau, \quad \forall t. \tag{1}$$

This system equation is suitable for representing stationary solutions such as equilibrium solutions or stationary periodic solutions. The parameter θ is a scaling of the size of the uncertainty in the system and we assume it belongs to an open interval I_θ, which contains 0.

We next summarize the assumptions on (1).

Assumption 1 *For the system in (1) we assume*

(*i*) *The nonlinearity $\varphi(\cdot)$ is C^1 (continuously differentiable).*
(*ii*) *For some exponential decay rate $\alpha > 0$ and all $\theta \in I_\theta$ (an open interval containing $\theta = 0$) we have $e^{\alpha t}h(t,\theta) \in \boldsymbol{L}_1[0,\infty)$ and furthemore that $h(t,\theta)$ is C^1 with respect to θ and has time differential $\mathrm{d}h(t,\theta) = \dot{h}_c(t,\theta)\mathrm{d}t + \sum_{k=0}^\infty h_k(\theta)\delta(t-t_k)\mathrm{d}t$, where $\delta(\cdot)$ denotes the dirac implulse, $e^{\alpha t}\dot{h}_c \in \boldsymbol{L}_1[0,\infty)$, $\sum_{k=0}^\infty e^{\alpha t_k}|h_k| < \infty$, $t_0 = 0$ and $t_k > 0$. Under these assumptions the Laplace transforms $H(s,\theta)$ and $sH(s,\theta)$ are (i) analytic in* $\mathrm{Re}\, s > -\alpha$, *(ii) continuous on $-\alpha + i\boldsymbol{R}$, and (iii) bounded such that for* $\mathrm{Re}\, s \geq -\alpha$ *we have* $\max(|sH(s,\theta)|, |H(s,\theta)|) \leq b$ *for some number b.*
(*iii*) *The system is called* nominal *when $\theta = 0$ and our assumption is that the nominal system has a T_0-periodic solution y_0. The periodic solution is called a* limit cycle *when it is isolated.*

We will derive conditions under which there remains a limit cycle when θ is perturbed from zero. Conditions for stability of the limit cycle are also derived.

A more concise operator notation for (1) is

$$y = H(s,\theta)\varphi(y). \tag{2}$$

The uncertain dynamics is often represented as a linear fractional transformation (LFT) (see Figure (1))

$$H(s,\theta) = H_{11}(s) + \theta H_{12}\Delta(s)(I - \theta H_{22}(s)\Delta(s))^{-1}H_{21}(s). \tag{3}$$

Here we normally assume that the nominal dynamics $H(s)$ is a finite dimensional transfer function with all poles in $\mathrm{Re}\, s < -\alpha$ and with H_{11} and either H_{12} or H_{21} strictly proper. If $\Delta(s)$ is a transfer function with impulse response function satisfying

$$\Delta(t) = \Delta_c(t) + \textstyle\sum_{k=0}^{\infty} \Delta_k \delta(t - t_k) \tag{4a}$$
$$e^{\alpha t}\Delta_c(t) \in \mathbf{L}_1[0\infty), \quad t_0 = 1,\ t_k > 0, \quad \textstyle\sum_{k=0}^{\infty} |e^{\alpha t_k}\Delta_k| < \infty \tag{4b}$$

then Assumption 1 holds for $I_\theta = (-\hat{\theta}, \hat{\theta})$ if the small gain condition $\hat{\theta}\|\Delta(s - \alpha)\|_{\mathbf{H}_\infty} \cdot \|H_{22}(s-\alpha)\|_{\mathbf{H}_\infty} < 1$ is satisfied.

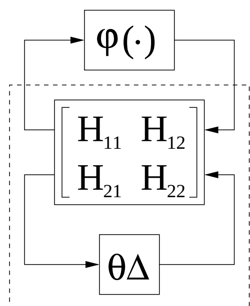

Fig. 1. Block diagram corresponding to the perturbed system in (2)-(3).

Example 1. Consider Van der Pol's equation with a dynamic uncertainty

$$\ddot{u}(t) + m(u(t)^2 - 1)\dot{u}(t) + u(t) = \theta(\Delta u)(t)$$

where $\Delta(s)$ is a transfer function with impulse response satisfying (4). To represent this system on the form (1) we introduce the new coordinates

$$\begin{aligned} x_1 &= -\dot{u} - m(u^3/3 - u) \\ x_2 &= u \end{aligned}$$

Differentiation gives

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\varphi(y(t)) + \theta B_\Delta(\Delta y)(t)\\ y(t) &= Cx(t)\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad B_\Delta = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and $\varphi(y) = -my^3/3 + (2+m)y$. Now the system can be represented on the LFT form in (2)-(3) with $\varphi(y) = -my^3/3 + (2+m)y$ and (where $C_\Delta = C$)

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B & B_\Delta \\ \hline C & 0 & 0 \\ C_\Delta & 0 & 0 \end{array}\right].$$

We will next discuss a condition from [3] for the existence of a periodic solution when θ is sufficiently close to 0. Later we derive conditions for local stability or more generally hyberbolicity of this limit cycle solution.

3 Existence of Solution

It is no restriction to assume that the period time $T_0 = 1$ since we can always re-scale the time axis by the transformation $t/T_0 \to t$, which gives the nominal dynamics

$$y_0(t) = \int_{-\infty}^{t} T_0 h(T_0(t-\tau), 0)\varphi(y_0(\tau))\mathrm{d}\tau, \qquad \text{for } t \in [0,1].$$

Hence, by redefining $T_0 h(T_0 t, 0) \to h(t,0)$ we can assume $T_0 = 1$. A general periodic solution to (1) can thus be written

$$y(t) = \int_{-\infty}^{t} T h(T(t-\tau), \theta)\varphi(y(\tau))\mathrm{d}\tau, \quad \text{for } t \in [0,1]. \tag{5}$$

The advantage of this reformulation is that a periodic solution can be represented as a pair $z = (y, T) \in C(1) \times \mathbf{R}$ of a 1-periodic trajectory and a period time. This will simplify our work considerably since the perturbation of the period time and the orbit are separated. In this section, the assumption is thus that (5) has a solution $z_0 = (y_0, 1)$ when $\theta = 0$. We often use the following concise notation for the system equation in (5)

$$y = H(s/T, \theta)\varphi(y) \tag{6}$$

where the nominal transfer function $H(s,0)$ in this paper often is assumed to have a finite dimensional state space realization.

With $Z = C(1) \times \mathbf{R}$ and $Y = C(1)$ we define the operator

$$F : Z \times I_\theta \to Y \quad \text{as} \quad (z,\theta) \mapsto F(z,\theta) = y - H(s/T,\theta)\varphi(y)$$

for any $z = (y,T) \in Z$. A solution to the equation $F(z_\theta,\theta) = 0$ corresponds to a periodic solution $z_\theta = (y_\theta, T_\theta) \in C(1) \times \mathbf{R}$ of (5) or similarly a T_θ-periodic solution $y_\theta(t/T_\theta)$ of (1).

We will use an implicit function theorem to derive conditions for the existence of a solution of the perturbed system. The Frechét derivative of F with respect to the trajectory at a periodic solution $z_\theta = (y_\theta, T_\theta)$ has the block structure

$$F'_z(z_\theta,\theta) = \begin{bmatrix} F'_y(z_\theta,\theta) & F'_T(z_\theta,\theta) \end{bmatrix} \tag{7a}$$

$$= \begin{bmatrix} I - L_{st}(z_\theta,\theta) & \frac{1}{T_\theta}(I - L_{st}(z_\theta,\theta))(t\dot{y}_\theta) \end{bmatrix} \tag{7b}$$

where

$$L_{st}(z_\theta,\theta) = H(s/T_\theta,\theta)\varphi'(y_\theta).$$

The notation L_{st} is used to indicate that this operator has to do with the stationary behavior of the system. The last component of the derivative follows after some calculation which is left to Appendix 1. Note that the argument $t\dot{y}_\theta$ of $I - L_{st}(z_\theta,\theta)$ does not belong to $C(1)$ while our claim is that the value does. The claim follows from the proof in Appendix 1.

It is interesting to note that the variational system corresponding to the nominal 1-periodic solution y_0 can be written $(I - L^0_{st})v = 0$, where $L^0_{st} = L_{st}(z_0, 0)$. The next proposition shows that $\dot{y}_0$ is in the kernel of $(I - L^0_{st})$, i.e. 1 is an eigenvalue of L^0_{st} with $\dot{y}_0$ as the corresponding eigenfunction. This follows since the periodic solution is unique only modulo arbitrary time translations.

Proposition 1. *We have* $\dot{y}_0 \in \operatorname{Ker}(I - L^0_{st})$, *where* $L^0_{st} = L_{st}(z_0, 0)$.

Proof. Let $h_0(t) = h(t,0)$. *By definition,*

$$y_0(t) = \int_{-\infty}^{t} h_0(t-\tau)\varphi(y_0(\tau))\mathrm{d}\tau$$

for all $t \in \boldsymbol{R}$. *Differentiation of this identity gives*

$$\begin{aligned}
\dot{y}_0(t) &= h_0(0)\varphi(y_0(t)) + \int_{-\infty}^{t} \mathrm{d}h_0(t-\tau)\varphi(y_0(\tau)) \\
&= h_0(0)\varphi(y_0(t)) + \lim_{T\to-\infty} \left[-h_0(t-\tau)\varphi(y_0(\tau))\right]_T^t \\
&\qquad + \int_{-\infty}^{t} h_0(t-\tau)\varphi'(y_0(\tau))\dot{y}_0(\tau)\mathrm{d}\tau \\
&= \int_{-\infty}^{t} h_0(t-\tau)\varphi'(y_0(\tau))\dot{y}_0(\tau)\mathrm{d}\tau
\end{aligned}$$

where we used that $h_0(t) \to 0$ *as* $t \to \infty$. *This concludes the proof.* □

Theorem 1. *If the operator $F_z'(z_0, 0)$ in (7) has a bounded right inverse then for each sufficiently small $|\theta|$ there exists $y_\theta \in C(1)$ and $T_\theta > 0$ that satisfies (5). The perturbed solution $y(\theta) = y_\theta$, $T(\theta) = T_\theta$ are C^1 functions of θ such that $y(0) = y_0$ and $T(0) = 1$. The solution is unique modulo time translation of $y_\theta(t)$.*

Proof. We sketch a proof. Let $\hat{F}(x, \theta) = F(z_0 + G_0 x, \theta)$, where G_0 is a bounded right inverse of $F_z'(z_0, 0)$. We have $\hat{F}(0, 0) = 0$ and $\hat{F}_x'(0, 0) = I$. By the implicit function theorem there exists a unique solution C^1 function $x_\theta := x(\theta)$ such that $\hat{F}(x_\theta, \theta) = 0$ for all sufficiently small $|\theta|$. This implies that $z_\theta = z_0 + G_0 x_\theta$ satisfies $F(z_\theta, \theta) = 0$. The only nonuniqueness is due to the choice of G_0 and it can be shown that this corresponds to a time translation. $\square$

We will next show how to construct a right inverse using the following lemma.

Lemma 1. *Let X be a normed vector space and consider a bounded linear operator $F : X \times \boldsymbol{R} \to X$ with block decomposition*

$$F = \begin{bmatrix} F_1 & f_2 \end{bmatrix}$$

where $F_1 = I - L$, with $L : X \to X$ being a compact operator with a simple eigenvalue at one and $f_2 \notin \operatorname{Im} F_1$ is a nonzero vector. Then a right inverse can be constructed as

$$F^\dagger = \begin{bmatrix} I \\ g \end{bmatrix} (F_1 + f_2 g)^{-1}$$

where $g \in X^*$ is any vector such that $(F_1 + f_2 g) : X \to X$ has a bounded inverse. In fact, any $g \in X^*$ such that $|g(e)| = |\langle e, g\rangle| > 0$ for a unit length vector $e \in \operatorname{Ker} F_1$ can be used.

Proof. The first claim follows immediately since

$$\begin{bmatrix} F_1 & f_2 \end{bmatrix} \begin{bmatrix} I \\ g \end{bmatrix} (F_1 + f_2 g)^{-1} = (F_1 + f_2 g)(F_1 + f_2 g)^{-1} = I.$$

Let g be defined as suggested in the second claim and suppose there exists $x \in X$ such that $(F_1 + f_2 g)x = 0$. If $y \in (\operatorname{Im} F_1)^\perp$ is nonzero, then

$$\langle (F_1 + f_2 g)x, y\rangle = \langle f_2, y\rangle \langle x, g\rangle = 0.$$

Since $\langle f_2, y\rangle \neq 0$ it follows that $\langle x, g\rangle = 0$. Hence, $x \in \operatorname{Ker} F_1 \cap \operatorname{Ker} g = \{0\}$. This shows that $\operatorname{Ker}(F_1 + f_2 g) = \{0\}$. Since $L - f_2 g$ is a compact operator and $\operatorname{Ker}(I - L + f_2 g) = \operatorname{Ker}(F_1 + f_2 g) = \{0\}$ it follows that $\operatorname{Im}(F_1 + f_2 g) = X$ (see e.g. Theorem 8.4-5 in [5]). Hence, $F_1 + f_2 g : X \to X$ is a bijection and it follows from Banach's isomorphism theorem that $(F_1 + f_2 g)^{-1}$ is a bounded operator. $\square$

We next use this lemma to construct a right inverse for the nominal operator.

Theorem 2. *Consider the operator $F'_z(z_0,0)$ defined in (7) in the finite dimensional case when $h(t,0) = Ce^{At}B\nu(t)$, where $\nu(t)$ is the unit step function and $A \in \boldsymbol{R}^{n\times n}$. Let us define*

$$A_{cl}(t) = A + B\varphi'(y_0(t))C, \qquad B_{cl}(t) = B\varphi'(y_0(t))$$

and let $x_0(t)$ be the 1-periodic solution of the nominal state space representation of (5)

$$\dot{x}_0(t) = Ax_0(t) + B\varphi(Cx_0(t)).$$

If $n-1$ of the characteristic multipliers of $A_{cl}(t)$ are different from 1 then $F'_z(z_0,0)$ has a bounded right inverse. One possible right inverse

$$F'_z(z_0,0)^{\dagger} = \begin{bmatrix} I + G_1 \\ G_2 \end{bmatrix} : C(1) \to C(1) \times \boldsymbol{R}$$

is defined as

$$(F'_z(z_0,0)^{\dagger}(w))(t) = (w(t) + \int_0^1 g_1(t,\tau)w(\tau)\mathrm{d}\tau, \int_0^1 g_2(1,\tau)w(\tau)\mathrm{d}\tau)$$

where

$$g_1(t,\tau) = \begin{cases} (\Gamma(t)\Phi_{cl}(1,t) + C)\Phi_{cl}(t,\tau)B_{cl}(\tau), & t > \tau \\ \Gamma(t)\Phi_{cl}(1,\tau)B_{cl}(\tau), & t < \tau \end{cases}$$
$$g_2(t,\tau) = k(I - \Phi_{cl}(1,0) - kk^T)^{-1}\Phi_{cl}(t,\tau)B_{cl}(\tau)$$

and

$$\Gamma(t) = C(\Phi_{cl}(t,0) + \dot{x}_0(t)k)(I - \Phi_{cl}(1,0) - kk^T)^{-1}.$$

Here $\Phi_{cl}(t,0)$ is the transition matrix corresponding to A_{cl} and $k = \dot{x}_0(0)^T$.

Proof. See Appendix 2. □

4 Stability

The system in (1) is generally of unknown or infinite dimension and the definition of stability needs extra care. We define local stability in terms of the variational system corresponding to the following *non-steady-state* version of (1)

$$y(t) = f(t) + \int_0^t h(t-\tau)\varphi(y(\tau))\mathrm{d}\tau, \quad t \geq 0 \tag{8}$$

where the dependence on θ is suppressed for notational convenience. In (8), $f(\cdot)$ represents initial conditions and external disturbances. The choice

$$f_0(t) = \int_{-\infty}^0 h(t-\tau)\varphi(y_0(\tau))\mathrm{d}\tau \tag{9}$$

gives the T_0-periodic solution $y_0(t)$, since (8) has a unique solution for any locally integrable function $f(\cdot)$. A linearization of (8) along the nominal periodic solution gives rise to the variational system

$$v = Lv + w \tag{10}$$

where we define

$$L : \mathbf{L}_2[0,\infty) \to \mathbf{L}_2[0,\infty) \quad \text{as} \quad v \mapsto Lv = H(s)\varphi'(y_0)v. \tag{11}$$

Note that we have defined the operator to act on $\mathbf{L}_2[0,\infty)$ and not $\mathbf{L}_\infty[0,\infty)$, which would be more natural for the linearization. However, this will allow an simple yet natural definition of stability in terms of the variational system (10) and the operator (11). Stability can also be defined in terms of the non-steady state system (8). This requires a more elaborate analysis but can be done, see [3].

The next proposition shows that the variational equation in (10) cannot be solved for arbitrary $w \in \mathbf{L}_2[0,\infty)$ unless $y_0 \equiv 0$. This follows because the input-output map $w \mapsto v$ defined by (10) is unbounded on $\mathbf{L}_2[0,\infty)$, since there is a finite energy input which maps to an infinite energy output. For finite dimensional systems this observation corresponds to the fact that the periodic linear system obtained as a result of linearization around a limit cycle always has a neutrally stable mode corresponding to a characteristic multiplier at unity.

Proposition 2. *If $y_0 \neq const$ is a T-periodic solution of (1) then*

$$w(t) = \int_{-\infty}^{0} h(t-\tau)\varphi'(y_0(\tau))\dot{y}_0(\tau)\mathrm{d}\tau$$

produces a periodic solution $v(t) = \dot{y}_0(t)$ of the variational system (10).

Proof. Let us differentiate $y_0(t)$. This gives

$$\begin{aligned}
\dot{y}_0(t) &= \frac{d}{dt}\int_{-\infty}^{t} h(t-\tau)\varphi(y_0(\tau))\mathrm{d}\tau = h(0)\varphi(y_0(t)) + \int_{-\infty}^{t} \mathrm{d}h(t-\tau)\varphi(y_0(\tau)) \\
&= h(0)\varphi(y_0(t)) + \lim_{T\to-\infty}\left[-h(t-\tau)\varphi(y_0(\tau))\right]_T^t \\
&\qquad + \int_{-\infty}^{t} h(t-\tau)\varphi'(y_0(\tau))\dot{y}_0(\tau)\mathrm{d}\tau \\
&= \int_{0}^{t} h(t-\tau)\varphi'(y_0(\tau))\dot{y}_0(\tau)\mathrm{d}\tau + w(t)
\end{aligned}$$

where we used that $\lim_{T\to-\infty} h(t-T)\varphi(y_0(T)) = 0$ since h is exponentially stable and continuous. □

In order to get around this problem we notice that the non-steady-state system, if stable, generally converges to $y_0(t+d)$, where $d \in \mathbf{R}$ is a nonzero *phase lag*. In fact, this is the reason for the neutrally stable mode of L, which implies that the image of the *return difference* $(I-L)$ has nonzero codimension. The lost term can be compensated for by considering the system

$$(I-L)v + ed = w \tag{12}$$

where $e = (I-L)(\dot{y}_0)$. Under Assumption 1 (ii) it can be shown that $\mathbf{L}_{2\alpha}[0,\infty) \ni e \notin \mathrm{Im}\,(I-L)$. The next step is to consider (12) as a system on the space of exponentially converging signals $\mathbf{L}_{2\alpha}[0,\infty)$. The neutral mode is now moved to the unstable and if the equation (12) can be proven to have an exponentially bounded solution for all exponentially bounded inputs then the limit cycle y_0 is said to be exponentially stable. We also consider the case when in addition to the neutral mode derived in Proposition 2 there are a finite number of unstable solutions of (10). We state this as a definition.

Definition 1. *If the system (12) has a unique solution $(v,d) \in \boldsymbol{L}_{2\alpha}[0,\infty) \times \boldsymbol{R}$ for all $w \in \boldsymbol{L}_{2\alpha}[0,\infty)$ then the limit cycle y_0 is called locally exponentially stable and α corresponds to the rate of exponential decay. Otherwise, if the subspace $W \subset \boldsymbol{L}_{2\alpha}[0,\infty)$ of codimension n_u is the largest subspace such that (12) has a unique solution $(v,d) \in \boldsymbol{L}_{2\alpha}[0,\infty) \times \boldsymbol{R}$ for all $w \in W$, then the limit cycle y_0 is said to have n_u unstable modes.*

This stability definition can be verified by computing the *stability defect* of the open loop operator L in (11). The stability defect is introduced as an equivalent of the notion "number of unstable closed-loop poles", which can be applied to time-varying systems. In the following definition, an open loop plant is represented by a linear operator on some normed space of signals.

Definition 2. *Let L be a bounded linear operator on a Banach space X, which is denoted $L \in \mathcal{L}(X,X)$. The feedback system with open loop operator L is called non-singular if there exists $\varepsilon > 0$ such that*

$$\|(I-L)u\| \geq \varepsilon \|u\|, \quad \forall u \in X. \tag{13}$$

The stability defect $\mathrm{def}(L)$ *of a non-singular system with the open loop operator L is defined as the codimension of the subspace*

$$\mathrm{Im}\,(I-L) = \{(I-L)u : u \in X\} \subset X.$$

*The stability defect, if well-defined on $X = \boldsymbol{L}_{2\alpha}$, will be called the α-*defect *of L in (11), denoted* $\mathrm{def}_\alpha(L)$.

The stability defect corresponds to the number of unstable modes of $(I-L)^{-1}$, i.e., the unstable closed loop poles, while condition (13) means that $I-L$ does not have zeros on the stability boundary. The motivation for working on $\mathbf{L}_{2\alpha}$ is that the neutrally stable mode in Proposition 2 is moved from the stability boundary to an unstable mode.

An important feature of the stability defect is the *zero exclusion principle*, which says that the stability defect $\mathrm{def}(L)$ remains constant as L changes continuously and condition (13) is satisfied. This is a robustness condition, which is used in the proof of Theorem 5.

Proposition 3 (Zero Exclusion Principle). *Let* $\mathfrak{L} = \{L \subset \mathcal{L}(X,X) : \exists \varepsilon > 0 \text{ s.t. } \|(I-L)u\| \geq \varepsilon\|u\|,\ \forall u \in X\}$. *Then any connected component* $\mathbf{L}$ *of* $\mathfrak{L}$ *containing an element with* $\mathrm{def}(L) < \infty$ *has constant stability defect, i.e., every* $\widetilde{L} \in \mathbf{L}$ *has* $\mathrm{def}(\widetilde{L}) = \mathrm{def}(L)$.

Proof. The non-singularity and the finite codimension of the image implies that L *is a Fredholm operator with index* $n = \mathrm{def}(L)$. *The proof follows since the set of Fredholm operators with constant finite index is open [4]. A proof is given in Appendix 3.* □

Theorem 3. *Suppose* $\mathrm{def}_\alpha(L) = n_u + 1$ *where* L *is defined in (11). Then* y_0 *is a hyperbolic solution with* n_u *unstable modes and the subspace* W *in Definition 1 is* $W = R \oplus P_{R^\perp}\mathrm{span}\{e\}$, *where* $e = (I-L)\dot{y}_0$, $R = \mathrm{Im}(I-L)$ *and* $P_{R^\perp}$ *is the orthogonal projection onto* $R^\perp$.

Proof. See Appendix 4. □

The next results shows that the stability defect is easy to compute in the finite dimensional case.

Theorem 4. *Consider the operator* L *defined in (11) in the finite dimensional case when* $h(t) = Ce^{At}B\nu(t)$, *where* $\nu(\cdot)$ *is the unit step function and* $\mathrm{Re}\lambda(A) < -\alpha$. *If the characteristic multipliers corresponding to* $A_{cl}(t) = A + B\varphi'(y_0(t))C$ *can be sorted as*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n_u}| > \lambda_{n_u+1} = 1 > |\lambda_{n_u+2}| \geq \cdots \geq |\lambda_n|$$

then $\mathrm{def}_\alpha(L) = n_u + 1$ *for* $\alpha \in (0, -\frac{\log|\lambda_{n_u+2}|}{T})$.

Proof. See Appendix 5. □

5 Main Result

By using Theorem 1 – Theorem 4 we obtain the following result.

Theorem 5. *Suppose the system in (1) has a* T_0*-periodic solution* y_0 *when* $\theta = 0$. *Assume further that the nominal system is finite dimensional with* $h(t) = Ce^{At}B\nu(t)$, *where* $\nu(\cdot)$ *is the unit step function and* $\mathrm{Re}\lambda(A) < -\alpha$. *If the characteristic multipliers corresponding to* $A_{cl}(t) = A + B\varphi'(y_0(t))C$ *can be sorted as*

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n_u}| > \lambda_{n_u+1} = 1 > |\lambda_{n_u+2}| \geq \cdots \geq |\lambda_n|$$

where $\alpha T < -\log(\lambda_{n_u+2})$, *then for all sufficiently small* $|\theta|$ *there exists a unique (modulo time translation) hyperbolic limit cycle solution with* n_u *unstable modes to equation (1).*

Proof. First note that the characteristic multipliers do not change if we normalize the nominal period time to $T_0 = 1$. *Existence of a solution in a neighborhood of* $\theta = 0$ *follows from Theorem 1 if* $F_z'(z_0, 0)$ *has a bounded right inverse. From Theorem 2, we see that this is the case since* $n-1$ *of the characteristic multipliers are different from* 1.

To prove the stability statement we consider the operator L *in (11), which becomes*

$$(L(\theta)v)(t) = \int_0^t T(\theta)h(T(\theta)(t-\tau),\theta)\varphi'(y_\theta(\tau))v(\tau)d\tau.$$

It follows from Theorem 4 that $L(0)$ *has* α*-defect* n_u+1. *From Proposition 3 we conclude that the* α*-defect remains constant for sufficiently small* $|\theta|$ *since* $L(\theta)$ *depends continuously on* θ. *Hence,* $\mathrm{def}_\alpha(L(\theta)) = n_u + 1$ *for sufficiently small* $|\theta|$, *which by Theorem 3 proves the statement on stability.* $\square$

Example 2. Theorem 5 shows that the characteristic multipliers of

$$A_{cl}(t) = A + B\varphi'(y_0(t))C = \begin{bmatrix} 0 & 1 \\ -1 & m(1-y_0(t)^2) \end{bmatrix}$$

must be sorted as $|\lambda_2| < \lambda_1 = 1$ in order for the limit cycle of the Van der Pol oscillator to be robustly stable. From Liouvilles formula we have

$$\lambda_2 = \det(\Phi_{cl}(1,0)) = e^{\int_0^1 \mathrm{tr}(A_{cl}(\tau))d\tau} = e^{\int_0^1 m(1-y_0(\tau)^2)d\tau}.$$

If, for example $m = 0.2$, then a numerical integration shows that $\lambda_2 = 0.34$ and the Van der Pol system thus has a robustly stable limit cycle for this value of m. This gives a new interpretation to the same condition in [2].

Appendix 1

We have

$$F_T'(z_\theta,\theta) = -\frac{s}{T_\theta^2}H_s'(s/T_\theta,\theta)\varphi(y_\theta)$$

which in time domain has the representation

$$\begin{aligned}(F_T'(z_\theta,\theta))(t) &= \int_{-\infty}^t h(T_\theta(t-\tau),\theta))\varphi(y_\theta(\tau))\mathrm{d}\tau \\ &\quad + \int_{-\infty}^t (t-\tau)T_\theta \mathrm{d}h(T_\theta(t-\tau),\theta))\varphi(y_\theta(\tau)).\end{aligned}$$

This is a $C(1)$ function by our assumptions on the transfer function $H(s,\theta)$ and since $\varphi(y_\theta(\tau)) \in C(1)$.

After a partial integration of the second term we get

$$\begin{aligned}(F_T'(z_\theta,\theta))(t) &= \int_{-\infty}^{t}(t-\tau)h(T_\theta(t-\tau),\theta)\varphi'(y_\theta(\tau))\dot{y}_\theta(\tau)\mathrm{d}\tau \\ &= -\int_{-\infty}^{t} h(T_\theta(t-\tau))\varphi'(y_\theta(\tau))(\tau\dot{y}_\theta(\tau))\mathrm{d}\tau \\ &\quad + th(0,\theta)\varphi(y_0(t)) + t\int_{-\infty}^{t} T_\theta \mathrm{d}h(T_\theta(t-\tau),\theta)\varphi(y_\theta(\tau)) \\ &= \frac{1}{T_\theta}\left(t\dot{y}_\theta(t) - \int_{-\infty}^{t} h(T_\theta(t-\tau))\varphi'(y_\theta(\tau))(\tau\dot{y}_\theta(\tau))\mathrm{d}\tau\right) \quad (14)\end{aligned}$$

where in the second equality we made a partial integration and the last equality follows because

$$\dot{y}_\theta(t) = T_\theta h(0,\theta)\varphi(y_\theta(t)) + \int_{-\infty}^{t} T_\theta^2 \mathrm{d}h(T_\theta(t-\tau),\theta)\varphi(y_\theta(\tau)).$$

A more concise formulation of (14) is $\frac{1}{T_\theta}(I - L_{st}(z_\theta,\theta))(t\dot{y}_\theta(t))$ which proves the statement.

Appendix 2

The operator $F_z'(z_0,0) : v \mapsto w$ has the following state space realization[3]

$$\begin{aligned}\dot{x} &= Ax + B\varphi'(y_0)v + \dot{x}_0\delta T \\ w &= v - Cx\end{aligned}$$

where $\dot{x}_0(t) = Ax_0(t) + B\varphi(y_0(t))$. In order to use Lemma 1 we identify $F_1 : v \mapsto w_1$ and $f_2 : gv \mapsto w_2$ as operators with the state space realizations

$$F_1 : \begin{cases} \dot{x}_1 = Ax_1 + B\varphi'(y_0)v \\ w_1 = v - Cx_1 \end{cases} \qquad f_2 : \begin{cases} \dot{x}_2 = Ax_2 + \dot{x}_0 gv \\ w_2 = -Cx_2. \end{cases}$$

Let $g : v \mapsto gv$ be defined by the state space realization

$$\begin{aligned}\dot{x}_3(t) &= Ax_3(t) + B\varphi'(y_0(t))v(t) + \dot{x}_0(t)kx_3(0) \\ gv &= kx_3(0).\end{aligned}$$

If $x_3(0) = x_1(0) + x_2(0)$ then $F_1 + f_2 g : v \mapsto w$ has the state space realization

$$\begin{aligned}\dot{x}_3(t) &= Ax_3(t) + B\varphi'(y_0(t))v(t) + \dot{x}_0(t)kx_3(0) \\ w(t) &= v(t) - Cx_3(t).\end{aligned}$$

The inverse of $F_1 + f_2 g$ can be derived by using $v = w + Cx_3$ in this equation. This gives the right inverse $F_z'(z_0,0)^\dagger : w \mapsto (v, \delta T)$

[3] All state equations in this section has a periodicity constraint of the form $x(1) = x(0)$ on the state vector. This is not written out explicitly.

$$\dot{x}_3(t) = (A + B\varphi'(y_0(t))C)x_3(t) + B\varphi'(y_0(t))w(t) + \dot{x}_0(t)kx_3(0) \quad \text{(15a)}$$
$$(v(t), \delta T) = (w(t) + Cx_3(t), kx_3(0)). \quad \text{(15b)}$$

In order for (15) to be well defined and bounded on $C(1)$ it is necessary and sufficient that the following equation has a solution for all $w \in C(1)$

$$x_3(0) = (\Phi_{cl}(1,0) + \dot{x}_0(0)k)x_3(0) + \int_0^1 \Phi_{cl}(1,\tau)B\varphi'(y_0(\tau),0)w(\tau)\mathrm{d}\tau$$

where we used that $\int_0^1 \Phi_{cl}(1,\tau)\dot{x}_0(\tau)d\tau kx_3(0) = \dot{x}_0(0)kx_3(0)$. Since we have $\mathrm{span}\{\dot{x}_0(0)\} = \mathrm{Ker}\,(I - \Phi_{cl}(1,0))$ it follows that there exists a vector k such that $I - \Phi_{cl}(1,0) - \dot{x}_0(0)k$ is invertible. Indeed, one possible choice is $k = \dot{x}_0(0)^T$. System (15) has the equivalent convolution form given in the theorem statement.

Appendix 3

The set $\mathbf{L}$ is open and by assumption connected. We will prove that the set of operators with constant (finite) stability defect is open. This proves the claim of the proposition since connectedness of $\mathbf{L}$ otherwise would be contradicted.

Consider an operator L with $\mathrm{def}(L) < \infty$. Since L is non-singular, we know that there exists $\varepsilon > 0$ such that $\|(I-L)u\| \geq \varepsilon\|u\|$ for all $u \in X$. Hence, it follows that $H = I - L$ has $\mathrm{Ker}\,H = 0$ and $\mathrm{codim}\,\mathrm{Im}H = \mathrm{def}(L)$. This means that H is a Fredholm operator with index

$$\mathrm{Ind}H := \dim \mathrm{Ker}\,H - \mathrm{codim}\,\mathrm{Im}\,H = -\mathrm{def}(L).$$

Since the codimension of $X_L = \mathrm{Im}\,H$ is finite it follows that there is a direct sum decomposition

$$X = X_L \oplus X_C$$

where $\dim X_C = \mathrm{def}(L)$. Now let ΔL be any perturbation of L with $\|\Delta L\| < \varepsilon/2$ and consider the maps $\widehat{H} : X \to X/X_C$ and $\widehat{\Delta L} : X \to X/X_C$ induced by H and ΔL. Here X/X_C denotes the quotient space and $\widehat{H} = q \circ H$, where $q : X \to X/X_C$ is the quotient map. Then $\|\widehat{\Delta L}\| < \varepsilon/2$ and $\widehat{H}$ has a bounded inverse by Banach's isomorphism theorem with norm bound $\|\hat{H}^{-1}\| \leq 1/\varepsilon$. We have

$$\widehat{H} - \widehat{\Delta L} = \widehat{H}(I - \widehat{H}^{-1}\widehat{\Delta L}),$$

from which it follows that $\hat{H} - \widehat{\Delta L}$ has a bounded inverse since $\|\widehat{H}^{-1}\widehat{\Delta L}\| \leq \|\widehat{H}^{-1}\| \cdot \|\widehat{\Delta L}\| < 1/2$. Hence, $\mathrm{Ind}(\widehat{H} - \widehat{\Delta L}) = 0$, which gives the relation

$$\mathrm{Ind}(\widehat{H} - \widehat{\Delta L}) = \dim X_C + \mathrm{Ind}(H - \Delta L) = 0 \quad (16)$$

since the quotient map $q : X \to X/X_C$ has index $\dim X_C$ and the index of the composite map $\widehat{H} - \widehat{\Delta L} = q \circ (H - \Delta L)$ is additive. Furthermore, invertibility of $\widehat{H} - \widehat{\Delta L}$ implies that $L + \Delta L$ is nonsingular and thus $\mathrm{Ker}\,(I - L - \Delta L) = 0$. Hence, from (16) we get

$$\mathrm{codim}\,\mathrm{Im}\,(I - L - \Delta L) = \dim X_C + \dim \mathrm{Ker}\,(I - L - \Delta L) = \dim X_C = \mathrm{def}(L)$$

which shows that $\mathrm{def}(L + \Delta L) = \mathrm{def}(L)$ for all $\|\Delta L\| < \varepsilon/2$.

Appendix 4

Let $R = \mathrm{Im}\,(I - L)$. We will prove that

$$(I - L)v + de = w, \qquad e = (I - L)(\dot{y}_0)$$

has a unique solution $(v, d) \in \mathbf{L}_{2\alpha}[0, \infty)$ if and only if $w \in W = R \oplus P_{R^\perp}\mathrm{span}\{e\}$. Here $P_{R^\perp} = I - P_R$, where P_R is the orthogonal projection onto $R = \mathrm{Im}\,(I - L) \subset \mathbf{L}_{2\alpha}[0, \infty)$. The assumptions on H in Assumption 1 can be used to prove that $e \in \mathbf{L}_{2\alpha}[0, \infty)$, see [3]. This implies that $R \subset W$ is a strict inclusion since $e \notin R$, i.e. $P_{R^\perp} e \neq 0$. We have

$$(I - P_R)((I - L)v + de) = d(I - P_R)e = (I - P_R)w$$

which gives $d = (I - P_R)w/((I - P_R)e)$ and the norm bound $|d| \leq c_1\|w\|_\alpha$, where $c_1 = 1/\|(I - P_R)e)\|_\alpha$. Using this d it follows that $(I - L)v = w - de$ has a unique solution in $\mathbf{L}_{2\alpha}[0, \infty)$ if and only if $w \in W$ because then $w - de \in R$. Since L is nonsingular (see definition of stability defect) there exists $\bar{c}_2$ such that

$$\|v\|_\alpha \leq \bar{c}_2\|w - de\|_\alpha \leq \bar{c}_2(1 + c_1\|e\|_\alpha)\|w\|_\alpha$$

and hence for each $w \in W$ we have found a unique solution satisfying the norm bound $\|v\|_\alpha^2 + |d|^2 \leq c\|w\|_\alpha^2$, where $c^2 = c_1^2 + c_2^2$ and $c_2 = \bar{c}_2(1 + c_1\|e\|_\alpha)$.

Appendix 5

For convenience we transform L to an equivalent operator L_α defined on $\mathbf{L}_2[0, \infty)$ as $L_\alpha = e_\alpha L e_\alpha^{-1}$ where e_α is defined by multiplication in the time domain with $e^{\alpha t}$. It can be shown that $\mathrm{def}(L_\alpha) = \mathrm{def}_\alpha(L)$, see [3]. We will show

(i) $\mathrm{Ker}\,(I - L_\alpha) = 0$
(ii) $\mathrm{codim}\,\mathrm{Im}\,(I - L_\alpha) = n_{u+1}$

Condition (i) and (ii) shows that L_α is a Fredholm operator with index $n_u + 1$. From Banach's isomorphism theorem it follows that $I - L_\alpha$ is nonsingular. This proves the theorem.

To prove (i) we assume there exists nonzero $v \in \mathbf{L}_2$ such that $(I - L_\alpha)v = 0$. In state space domain this means that

$$\dot{x} = (A + \alpha I)x + B\varphi'(y_0)v,\ x(0) = 0$$
$$0 = v - Cx$$

which implies that $v = Cx$ and $\dot{x} = (A + \alpha I + B\varphi'(y_0)C)x,\ x(0) = 0$. This contradicts the assumption that v is nonzero. Hence, $\mathrm{Ker}\,(I - L_\alpha) = 0$.

To prove (ii) we use $(\mathrm{Im}\,(I - L_\alpha))^\perp = \mathrm{Ker}\,(I - L_\alpha)^*$. One possible state space representation of the adjoint system $v \mapsto w = (I - L_\alpha^*)v$ is

$$\dot{x} = -(A + \alpha I)^T x + C^T v,\ x(\infty) = 0$$
$$w = v + \varphi'(y_0)^T B^T x.$$

Any $v \in \mathrm{Ker}\,(I - L_\alpha^*)$ must satisfy $v = -\varphi'(y_0)^T B^T x$ where

$$\dot{x} = -(A + \alpha + B\varphi'(y_0)C)^T x, \quad x(\infty) = 0. \tag{17}$$

A result by Lyapunov shows that there exists a time-periodic coordinate transformation that turns system (17) into a linear system with constant coefficients [2]. It is no restriction to assume the new coordinates are chosen such that

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad \begin{bmatrix} z_1(\infty) \\ z_2(\infty) \end{bmatrix} = 0$$

where $A_1 \in \mathrm{C}^{(n_u+1)\times(n_u+1)}$ is stable and A_2 is unstable with $|\mathrm{eig}(e^{A_2 T})| \geq |e^{-\alpha T}/\lambda_{n_u+2}| > 1$. If the coordinates are related as

$$x(t) = \begin{bmatrix} P_1(t) \ P_2(t) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$$

where $P(t) = \begin{bmatrix} P_1(t) \ P_2(t) \end{bmatrix}$ is invertible and T periodic, then we see that

$$\mathrm{Ker}\,(I - L_\alpha)^* = \left\{ v(t) = -\varphi'(y_0(t))^T B^T P_1(t) e^{A_1 t} z_1(0) : z_1(0) \in \mathbf{R}^{n_u+1} \right\}.$$

This is an $n_u + 1$ dimensional space.

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