Local Robustness of Oscillator Networks

Ulf Jönsson *†

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Abstract

Conditions for robust stability of networks of identical coupled oscillators are derived. It is shown that local stability and robustness can be verified using the characteristic multipliers of a variational system obtained by linearizing along the synchronized network solution. The computation of a robustness margin is considered using small gain conditions. All analysis conditions decompose to lower dimensional problems determined by the spectrum of the network interconnection matrix and the dynamics of the individual oscillators. It is also discussed how the network topology can be analyzed and designed for robustness and performance.

1 Introduction

The theory and application of oscillator networks have in recent years received ample attention in applied mathematics, biology, and engineering. We refer to the surveys in [5,12] and the recent contributions in [9,13,14,17]. Many of these works discuss stability and synchronization of the networks. Robustness properties have not been discussed explicitly but they appear implicitly in several works. Some examples are the conditions for the onset of oscillation in [10,11] and the synchronization results in [7,8], which all are

*This work was supported by the Swedish Research Council for Engineering Sciences.

†The author is with th Division of Optimization and Systems Theory, Royal Institute of Technology, SE 10044 Stockholm, Sweden. ulfj@math.kth.se

developed using dissipation theory. Robustness to the removal of nodes in the network is discussed in e.g. [15].

In this paper we consider robust stability of networks of identical coupled oscillators subject to dynamic uncertainty in the individual oscillators. We assume that the coupling of the network is such that the limit cycle of a single oscillator is embedded in the network solution. When this is the case the network is said to be synchronized and all oscillators remain stable in their nominal limit cycle. We provide conditions for the system to remain near synchronized when the oscillators are perturbed and no longer identical. For the robustness result we use a recent result in [4], which will be reviewed in the next section. There it is proven that a well-known condition for robustness of limit cycles in finite dimensional systems extends to a class of systems with dynamic uncertainties. In this way a condition on the variational system (obtained after linearization along the limit cycle) is used to verify that the limit cycle persist and is stable for all sufficiently small dynamic perturbations of the system. We consider symmetric networks where the robustness condition decomposes to stability conditions on the characteristic multipliers of a number of low dimensional systems, one for each spectral value of the interconnection matrix. This is analogous to the results in [6, 18] with the exception that we obtain a new robustness interpretation of this condition.

Bounds on a robustness margin can be estimated using a number of small gain conditions developed in [4]. We apply this result to a symmetric interconnection structure. Our analysis shows that the network cannot have better robustness margin than an individual oscillator. This is expected, but what is more interesting is that small gain analysis provides insight into how the network topology should be designed to make the modes corresponding to unsynchronized states as robust as possible with as fast rate of convergence as possible.

We illustrate design of the network on the simpler problem of optimizing the rate at which the network synchronizes. The dependence of the synchronization rate on the network size is also discussed. We end the paper with a discussion on robustness to perturbation of the network topology and its dependence on the size of the network. A brief version of the paper appeared in the IEEE conference [2]

Notation

We let $\mathbf{L}_2(1)$ denote the space of square integrable 1-periodic functions and $C(1)$ is the set of continuous 1-periodic functions equipped with the norm $\|v\|_{C(1)} = \sup_{t\in[0,1]} |v(t)|$, where $|\cdot|$ always denotes the Euclidean norm. We let $\mathbf{1} = \begin{bmatrix}1 & \dots & 1\end{bmatrix}^T$ and $\otimes$ denotes the Kronecker product. The spatial norm will always be the Euclidean norm $|v| = (\sum_{i=1}^n v_i^2)^{1/2}$. We will almost always denote norms and induced norms on $\mathbf{L}_2(1)$ without the suffix, i.e. $\|\cdot\| := \|\cdot\|_{\mathbf{L}_2(1)}$ and $\|\cdot\| := \|\cdot\|_{\mathbf{L}_2(1)\to\mathbf{L}_2(1)}$. At several places we consider the space $C(1)\times\mathbf{R}$ with the norm $\|(v,T)\|_{C(1)\times\mathbf{R}} = (\|v\|_{C(1)}^2 + |T|^2)^{1/2}$ and similarly for $\mathbf{L}_2(1)\times\mathbf{R}$.

2 Robustness of Single Oscillators

We consider systems consisting of a feedback interconnection of an exponentially stable linear time-invariant (LTI) plant and a memoryless nonlinearity

$$y(t) = \int_{-\infty}^{t} h(t-\tau,\theta)\varphi(y(\tau))d\tau, \quad \forall t \tag{1}$$

This system equation is suitable for representing stationary solutions such as equilibrium solutions or stationary periodic solutions. The parameter θ is a scaling of the size of the uncertainty in the system.

We next summarize the assumptions on (1).

Assumption 1. For the system in (1) we assume

(i) The nonlinearity $\varphi(\cdot)$ is C^1 (continuously differentiable).

(ii) For some exponential decay rate $\alpha > 0$ and all $\theta \in I_\theta$ (an open interval containing $\theta = 0$) we have $e^{\alpha t}h(t,\theta) \in \mathbf{L}_1[0,\infty)$ and furthermore that $h(t,\theta)$ is C^1 with respect to θ and has time differential $dh(t,\theta) = \dot{h}_c(t,\theta)dt + \sum_{k=0}^{\infty} h_k(\theta)\delta(t-t_k)dt$, where $\delta(\cdot)$ denotes the Dirac impulse, $e^{\alpha t}\dot{h}_c \in \mathbf{L}_1[0,\infty)$, $\sum_{k=0}^{\infty} e^{\alpha t_k}|h_k| < \infty$, $t_0 = 0$ and $t_k > 0$. Under these assumptions the Laplace transforms $H(s,\theta)$ and $sH(s,\theta)$ are (i) analytic in $\operatorname{Re} s > -\alpha$, ($ii$) continuous on $-\alpha + i\mathbf{R}$, and (iii) bounded such that for $\operatorname{Re} s \geq -\alpha$ we have $\max(|sH(s,\theta)|, |H(s,\theta)|) \leq b$ for some number b.

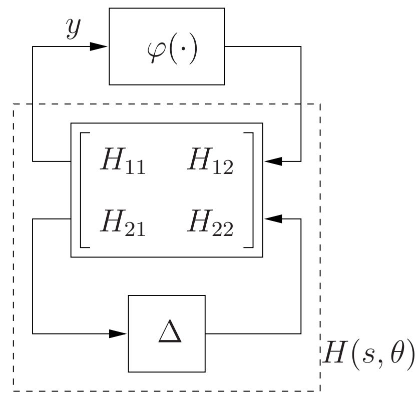

Figure 1: Block diagram corresponding to the perturbed system in (2)-(3).

(*iii*) The system is called *nominal* when $\theta = 0$ and our assumption is that the nominal system has a T_0-periodic solution y_0. The periodic solution is called a *limit cycle* when it is isolated.

In this section we discuss results from [3,4] that provide conditions under which there remains a stable limit cycle when θ is perturbed from zero.

A more concise operator notation for (1) is

$$y = H(s,\theta)\varphi(y) \tag{2}$$

The uncertain dynamics is often represented as a linear fractional transformation (LFT) (see Figure (1))

$$H(s,\theta) = H_{11}(s) + H_{12}\Delta(s,\theta)(I - H_{22}(s)\Delta(s,\theta))^{-1}H_{22}(s) \tag{3}$$

where we assume

1. $\Delta(s,0) = 0$ and Δ is C^1 as a function of θ
2. the nominal dynamics and $\Delta(s,\theta)$ is such that the $H(s,\theta)$ is strictly proper and exponentially stable for all $\theta \in I_\theta = (-\hat{\theta},\hat{\theta})$

This is often easy to verify using analysis techniques from robust control.

Example 1. Consider Van der Pol's equation with a dynamic uncertainty

$$\ddot{y}(t) + m(y(t)^2 - 1)\dot{y}(t) + y(t) = (\Delta(s,\theta)y)(t).$$

An interesting case is $\Delta(s,\theta) = e^{-s\theta} - 1$, which corresponds to a delay in the system equation, i.e. $\ddot{y}(t) + m(y(t)^2 - 1)\dot{y}(t) + y(t-\theta) = 0$.

To represent this system on the form (1) we introduce the new coordinates

$$\begin{aligned} x_1 &= -\dot{y} - m(y^3/3 - y) \\ x_2 &= y \end{aligned}$$

Now the system can be represented on the LFT form in (2)-(3) with $\varphi(y) = -my^3/3 + (2+m)y$ and

$$H(s) = \left[\begin{array}{c|cc} A & B & B_\Delta \\ \hline C & 0 & 0 \\ C_\Delta & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & -1 \\ -1 & -2 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

Local Results from [3, 4]

We will next discuss a local robustness result in the case when the nominal dynamics of (1) is finite dimensional, i.e. $H(s,0) = C(sI - A)^{-1}B$, where A is Hurwitz. Although the nominal dynamics is finite dimensional the perturbed system is generally of unknown or infinite dimension and the definition of stability needs extra care. Local stability is defined in terms of the variational system corresponding to the following *non-steady-state* version of (1)

$$y(t) = f(t) + \int_0^t h(t-\tau)\varphi(y(\tau))d\tau, \quad t \geq 0 \tag{4}$$

where the dependence on θ was suppressed for notational convenience. In (4), $f(\cdot)$ represents initial conditions and external disturbances. The choice

$$f_0(t) = \int_{-\infty}^0 h(t-\tau)\varphi(y_0(\tau))d\tau \tag{5}$$

gives the T-periodic solution $y_0(t)$. A linearization of (4) along the nominal periodic solution gives rise to the variational system

$$v = Lv + w \tag{6}$$

where $L : \mathbf{L}_2[0,\infty) \to \mathbf{L}_2[0,\infty)$ is defined as

$$v \mapsto Lv = H(s)\varphi'(y_0)v \tag{7}$$

We here let the linearized dynamics operate on $\mathbf{L}_2[0,\infty)$. The variational equation in (6) cannot be solved for arbitrary $w \in \mathbf{L}_2$ unless $y_0 \equiv 0$. This follows because it can be shown that the input-output map $w \mapsto v$ defined by (6) is unbounded on $\mathbf{L}_2$, since the input

$$w(t) = \int_{-\infty}^{0} h(t-\tau)\varphi'(y_0(\tau))\dot{y}_0(\tau)d\tau$$

produces a periodic solution $v(t) = \dot{y}_0(t)$, i.e. this finite energy input maps to an infinite energy output. For finite dimensional systems this observation corresponds to the fact that the periodic linear system obtained as a result of linearization around a limit cycle always has a *neutrally stable* mode corresponding to an eigenvalue at unity. In order to get around this problem we notice that the non-steady-state system generally converges to $y_0(t+d)$, where $d \in \mathbf{R}$ is a nonzero *phase lag*. In fact, this is the reason for the neutrally stable mode of L, which in turn implies that the *return difference* $(I-L)$ has nonzero co-dimension. The lost term in the image space can be compensated for by considering the system

$$(I-L)v + ed = w \tag{8}$$

where $e = (I-L)(\dot{y}_0) \neq \mathrm{Im}\,(I-L)$.

The next step is to consider (8) as a system on the space of exponentially converging signals

$$\mathbf{L}_{2\alpha} = \left\{ v \in \mathbf{L}_2[0,\infty) : \int_0^\infty e^{2\alpha t}|v(t)|^2 dt < \infty \right\}$$

with norm $\|v\|_\alpha^2 = \int_0^\infty e^{2\alpha t}|v(t)|^2 dt$. The neutral mode is now moved to the unstable and if the equation can be proven to have an exponentially bounded solution for all exponentially bounded inputs then the limit cycle y_0 is said to be exponentially stable. We state this as a definition.

Definition 1. The limit cycle y_0 is called locally exponentially stable if there exists $c > 0$ such that for all $w \in \mathbf{L}_{2\alpha}[0,\infty)$ there exists a unique solution $(v,d) \in \mathbf{L}_{2\alpha}[0,\infty) \times \mathbf{R}$ to (8), which is bounded as

$$\|v\|_\alpha^2 + |d|^2 \leq c\|w\|_\alpha^2$$

To verify stability we need to compute the *stability defect* defined next. in (7)

Definition 2. Let $L : \mathbf{L}_{2\alpha}[0,\infty) \rightarrow \mathbf{L}_{2\alpha}[0,\infty)$ be defined as in (7). The feedback system in (8) with *open loop operator* L is called *non-singular* if there exists $\varepsilon > 0$ such that

$$\|(I-L)v\| \geq \varepsilon\|v\|, \quad \forall v \in \mathbf{L}_{2\alpha}[0,\infty) \tag{9}$$

The *stability defect* $\mathrm{def}_\alpha(L)$ of a non-singular system with the open loop operator L is defined as the co-dimension of the subspace

$$\mathrm{Im}\,(I-L) = \{(I-L)u : u \in \mathbf{L}_{2\alpha}\} \subset \mathbf{L}_{2\alpha}.$$

The stability defect corresponds to the number of unstable modes of $(I-L)^{-1}$, i.e., the unstable closed loop poles, while condition (9) means that $I-L$ does not have zeros on the stability boundary. It can be verified that y_0 is exponentially stable if and only if $\mathrm{def}_\alpha(L) = 1$. The motivation for working on $\mathbf{L}_{2\alpha}$ is that the neutrally stable mode of L is moved to an unstable mode. The non-singularity assumption implies, by the robustness of the Fredholm index, that the stability properties are preserved under small perturbations of the operator L.

The next result shows that the characteristic multipliers corresponding to a nominal finite dimensional linearized dynamics determine conditions under which the nominal limit cycle persists under small enough perturbations. They also determine the stability defect of L and thus the number of unstable modes. We use that the *characteristic multipliers* of a periodic matrix $A(t) = A(t+T_0)$ are the eigenvalues of the monodromy matrix $\Phi(T_0, 0)$, where

$$\frac{d}{dt}\Phi(t,0) = A(t)\Phi(t,0), \qquad \Phi(0,0) = I$$

Theorem 1. [4] Suppose Assumption 1 holds and consider the case when $h(t,0) = Ce^{At}B\nu(t)$, where $\nu(\cdot)$ is the unit step function. If the characteristic multipliers corresponding to $A_{cl}(t) = A + B\varphi'(y_0(t))C$ can be sorted as

$$1 = |\rho_1| > |\rho_2| \geq \cdots \geq |\rho_n|$$

then there exists a unique (modulo time translation) exponentially stable limit cycle solution to (1) for all sufficiently small θ. The perturbed solution and its period time $y(\theta)$, $T(\theta)$ are C^1 functions of θ such that $y(0) = y_0$ and $T(0) = T_0$.

Robustness Margin

We will next consider the computation of bounds on the size of perturbation the limit cycle of (1) can tolerate. The introduction of uncertainty in the system dynamics perturb both the period time and the orbit of the limit cycle which is in stark contrast to the traditional problems in robust control where the equilibrium solution remains fixed when the system is perturbed, see e.g. [19]. By rescaling the time axis and the system dynamics it is possible to represent general T_θ-periodic solutions of (1) as

$$y_\theta = H(s/T_\theta, \theta)\varphi(y_\theta)$$

where y_θ is a 1-periodic trajectory and T_θ is the period time. If we let $z_\theta = (y_\theta, T_\theta)$ be a trajectory and period time pair for a periodic solution of (1) then we can define robustness margin as follows.

Definition 3 *(Robustness Margin).* A bound $\bar{\theta}$ is a robustness margin for the nominal solution y_0 of (1) if for a given tolerance $r_0 > 0$ there exists a unique (modulo time translation) C^1 mapping $z_\theta : [-\bar{\theta}, \bar{\theta}] \to \mathcal{Z}$, where z_θ is an exponentially stable solution of (1) for all $\theta \in [-\bar{\theta}, \bar{\theta}] \subset I_\theta$, and

$$\mathcal{Z} = \{(y, T) \in C(1) \times \mathbf{R} : \|y - y_0\|^2_{C(1)} + |T - 1|^2 \leq r_0^2\}$$

To obtain a robustness result we need to verify existence of a solution in $\mathcal{Z}$ as well as stability. For the stability condition we use the notation

$$L(z_\theta, \theta) = H(s/T_\theta, \theta)\varphi'(y_\theta)$$

and $L_0 := L(z_0, 0)$. Condition (i) in Theorem 2 below ensures $\text{def}_\alpha(L_0) = 1$ while (ii) is a zero exclusion result showing that also $\text{def}_\alpha(L(z_\theta, \theta)) = 1$, $\forall z_\theta \in \mathcal{Z},\ \theta \in [-\bar{\theta}, \bar{\theta}]$, which implies that all perturbed limit cycles are stable. For existence we introduce the operators $F : (C(1) \times \mathbf{R}) \times I_\theta \mapsto C(1)$ defined as

$$F(z_\theta, \theta) = y_\theta - H(s/T_\theta)\varphi(z_\theta)$$

By assumption $F(z_0, 0) = 0$ and we want to verify that there exists $z_\theta \in \mathcal{Z}$ such that $F(z_\theta, \theta) = 0$ for all $\theta \in [-\bar{\theta}, \bar{\theta}]$. An implicit function theorem is used to do this. We let the Frechèt derivative

$$\begin{aligned} F'_z(z_\theta, \theta) &= \begin{bmatrix} F'_y(z_\theta, \theta) & F'_T(z_\theta, \theta) \end{bmatrix} \\ &= \begin{bmatrix} I - L_{st}(z_\theta, \theta) & \frac{1}{T_\theta}(I - L_{st}(z_\theta, \theta))(t\dot{y}_\theta) \end{bmatrix} \end{aligned}$$

where

$$L_{st}(z_\theta, \theta) = H(s/T_\theta, \theta)\varphi'(y_\theta)$$

act as an operator on $\mathbf{L}_2(1)$. The distinction between L and L_{st} is that the later has to do with the stationary behavior of the system while the former has to do with the transient behavior of the system. Condition (i) in Theorem 2 below ensures that the right inverse $F'_z(z_0, 0)^\dagger$ is well defined and bounded, which by an implicit theorem proves local existence. Condition (iii) and (iv) are used to prove that there exists $z_\theta \in \mathcal{Z}$ for all $\theta \in [-\bar{\theta}, \bar{\theta}]$.

Theorem 2. [4] Suppose $h(t, 0) = Ce^{At}B\nu(t)$, where $\nu(\cdot)$ is the unit step function. If

(i) the characteristic multipliers of $A_{cl}(t) = A + B\varphi'(y_0(t))C$ can be sorted as

$$1 = \rho_1 > |\rho_2| \geq |\rho_3| \geq \ldots \geq |\rho_n|$$

(ii) there exists $0 < \epsilon < c$ such that

$$\|(I - L_0)v\|_\alpha \geq c\|v\|_\alpha, \quad \forall v \in \mathbf{L}_{2\alpha}$$
$$\sup_{z\in\mathcal{Z}, |\theta|\leq\bar{\theta}} \|L(z_\theta, \theta) - L_0\|_\alpha \leq c - \epsilon$$

(iii) $\displaystyle\sup_{z\in\mathcal{Z}, |\theta|\leq\bar{\theta}} \|F'_z(z, \theta) - F'_z(z_0, 0)\| \cdot \|F'_z(z_0, 0)^\dagger\| < 1$

(iv) for $\Delta(z, \theta) = F'_z(z, \theta) - F'_z(z_0, 0)$ we have

$$\sup_{z\in\mathcal{Z}, |\theta|\leq\bar{\theta}} \|F'_z(z_0, 0)^\dagger(I - \Delta(z, \theta)F'_z(z_0, 0)^\dagger)^{-1}F'_\theta(z, \theta)\|_{\mathbf{L}_2(1)\to C(1)\times\mathbf{R}} < \frac{r_0}{\overline{\theta}}$$

then $\overline{\theta}$ is a robustness margin for the solution y_0.

3 Network of Oscillators

Here we consider a network of coupled oscillators. Each individual oscillator is assumed to have the dynamics

$$y_k(t) = \int_{-\infty}^{t} h_k(t - \tau, \theta)\varphi(y_k(\tau), 0)d\tau \tag{10}$$

Each impulse response function h_k corresponds to an uncertain strictly proper and exponentially stable transfer function $H_k(s, \theta)$. We assume that the nominal transfer functions are identical and satisfies Assumption 1. We further assume that $\|H_k(s, \theta) - H_k(s, 0)\| \le \gamma|\theta|$ for all k, i.e. the systems have the same uncertainty bounds. The oscillators may not be identical when $\theta \neq 0$, they only have the same uncertainty bound.

A compact description of the system equation in (10) is $y_k = H_k(\theta)\varphi(y_k, 0)$. The additional argument of the nonlinearity is used for interconnection of N oscillators of the above form

$$
\begin{aligned}
y_k &= H_k(\theta)\varphi(y_k, \bar{y}_k) \\
\bar{y}_k &= \sum_{l \neq k} \gamma_{k,l}(y_l - y_k)
\end{aligned}
\tag{11}
$$

If we let

$$
y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \vdots \\ \bar{y}_N \end{bmatrix}, \quad \Phi(y, \bar{y}) = \begin{bmatrix} \varphi(y_1, \bar{y}_1) \\ \vdots \\ \varphi(y_N, \bar{y}_N) \end{bmatrix}
$$

$$
\Gamma = \begin{bmatrix} -\sum_{l \neq 1} \gamma_{1,l} & \gamma_{1,2} & \cdots & \gamma_{1,N} \\ \gamma_{2,1} & -\sum_{l \neq 2} \gamma_{2,l} & \cdots & \gamma_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N,1} & \gamma_{N,2} & \cdots & -\sum_{l \neq N} \gamma_{N,l} \end{bmatrix}
$$

$$
H(\theta) = \text{diag}\,(H_1(\theta), \dots, H_N(\theta)),
$$

then the system equation can be rewritten as

$$
y = H(\theta)\Phi(y, \Gamma y) \tag{12}
$$

We make the standing assumption that Γ is symmetric. The network is called nominal when $\theta = 0$ and we have assumed that each individual nominal oscillator has a stable T_0-periodic limit cycle y_0. Since $0 \in \text{eig}(\Gamma)$ with corresponding eigenvector $\mathbf{1}$, this implies that $y_k = y_0$, $k = 1, \dots, N$ is a solution of the nominal network. We say that the network *synchronize* if this solution is stable.

The next proposition shows that if the individual oscillators are locally robust and if the network design is appropriate then the network is robust in the sense that a near synchronized solution remains after the individual oscillators are perturbed by independent but sufficiently small dynamic perturbations.

Proposition 1. Suppose the nominal dynamics of each individual oscillator in (11) is finite dimensional with $H_k(s,0) = C(sI - A)^{-1}B$, where A is Hurwitz. Suppose all but one characteristic multiplier of

$$A_k(t) = A + B\left(\varphi'_y(y_0(t),0) + \lambda_k \varphi'_{\bar{y}}(y_0(t),0)\right)C, \qquad \lambda_k \in \mathrm{eig}(\Gamma) \tag{13}$$

are strictly inside the unit circle. Then the network in (11) is robust in the sense that for all sufficiently small θ, there exists a locally exponentially stable solution $y_k(\theta)$, $k = 1, \ldots, N$ to the network. The perturbed solution and its period time $y_k(\theta)$, $T_k(\theta)$ are C^1 functions of θ such that $y_k(0) = y_0$ and $T_k(0) = T_0$.

Proof. The variational system corresponding to the nominal dynamics has the system matrix

$$A_{cl}(t) = I \otimes (A + B\varphi'_y(y_0(t),0)C) + (I \otimes B\varphi'_{\bar{y}}(y_0(t),0))(\Gamma \otimes I)(I \otimes C)) \tag{14}$$

which can be simplified to

$$A_{cl}(t) = I \otimes (A + B\varphi'_y(y_0(t),0)C) + \Gamma \otimes (B\varphi'_{\bar{y}}(y_0(t),0)C)$$

Then $\dot{\bar{x}}_0(t) = \mathbf{1} \otimes \dot{x}_0$ is the eigenfunction corresponding to the simple characteristic multiplier at 1. Further if v_k is an eigenvector corresponding to $\lambda_k \in \mathrm{eig}(\Gamma)$ and $x_k(t)$ is an eigenfunction corresponding to the characteristic multiplier ρ_k of $A_k(t)$, then $v_k \otimes x_k(t)$ is an eigenfunction corresponding to the characteristic multiplier ρ_k of $A_{cl}(t)$. Hence, the result follows from Theorem 1 in the previous section. $\square$

Example 2. Consider the case when

$$\ddot{y}_k + m((y_k + \bar{y}_k)^2 - 1)(\dot{y}_k + \dot{\bar{y}}_k) - 2\dot{\bar{y}}_k + y_k = \theta \Delta_k y_k$$

where $\bar{y}_k = -y_k + \dfrac{1}{N}\sum_{l \neq k} y_l$. Using the new states

$$\begin{aligned} x_{1,k} &= -\dot{y}_k - m((y_k + \bar{y}_k)^3/3 - y_k - \bar{y}_k) + 2\bar{y}_k \\ x_{2,k} &= y_k \end{aligned}$$

gives the system equation

$$
\begin{aligned}
\dot{x}_k &= Ax_k + B_1\varphi(y_k, \bar{y}_k) + \theta B_\Delta(\Delta y_k)(t)\\
y_k &= Cx_k
\end{aligned}
$$

where $\varphi(y, \bar{y}) = \phi(y + \bar{y})$ and $\phi(y) = -my^3/3 + (2+m)y$, A, B_1, B_Δ and C are defined as in Example 1. Proposition 1 shows all but one characteristic multipliers of

$$
A_k(t) = A + (1+\lambda_k)B\varphi'(y_0(t))C
$$

must be strictly inside the unit disc in order for the Van der Pol oscillator network to be robustly stable. We have $1 + \lambda_1 = 1$ and $1 + \lambda_k = -1/(N-1)$ for the remaining eigenvalues. If, for example $m = 0.2$, then Liouvilles formula gives for $k = 1$ that the characteristic multipliers are 1 and $\rho = e^{\int_0^1 \mathrm{tr}(A_1(\tau))d\tau} = e^{\int_0^1 m(1-y_0(\tau)^2)d\tau} = 0.34$. Numerical calculation shows that the remaining characteristic multipliers are within the unit circle for all network size, i.e., for all integers N. This shows that the Van der Pol oscillators in an all-to-all coupling are robust.

Robustness Margin

We have the following basic property.

Proposition 2. The network in (11) cannot have better robustness margin than the individual oscillators in (10).

Proof. See the appendix. □

The intuition behind the proposition is that the system effectively is decoupled when it is synchronized and in this sense the network is no more robust than the individual oscillator. Still the network structure affects the robustness of the synchronization process. We will use Theorem 2 to derive design rules for the coupling matrix. We consider the special case when the network in (12) has the special form

$$
y = H(\theta)\Phi(y + \bar{y}) \tag{15}
$$

where each individual oscillator has an additive uncertainty $H_k(\theta) = H_0 + \theta\Delta_k$ and nonlinearity $\varphi(y_k, \bar{y}_k) = \varphi(y_k + \bar{y}_k)$, i.e., the nonlinearity has only one input. Furthermore $\bar{y}_k = \sum_{l \neq k} \gamma_{k,l}(y_l - y_k)$ with a coupling matrix $\Gamma = \Gamma^T = [\gamma_{k,l}]_{k,l=1}^N$ of the same structure as before. We assume without loss of generality that the nominal limit cycle has period $T_0 = 1$.

We consider the different conditions in Theorem 2.

(i) The condition on the characteristic multipliers is completely analogous to Proposition 1.

(ii) For the robust stability condition we first consider the computation of the index of non-singularity ($c > 0$) as

$$\|(I - L_0)v\|_\alpha \geq c\|v\|_\alpha, \ \forall v \in \mathbf{L}_{2\alpha}[0, \infty)$$

Using that $L_0 = (I + \Gamma) \otimes H_0\varphi'(y_0(t))$ we will obtain a simple expression. If we let v_k be a unit length eigenvector corresponding to the eigenvalue $\lambda_k \in \text{eig}(\Gamma)$ then with $v = v_k \otimes v_i$ we get

$$\begin{aligned}\|(I - L_0)v\|_\alpha &= \|v_k \otimes (I - (1 + \lambda_k)H_0\varphi'(y_0(t)))v_i\|_\alpha \\ &= \|(I - (1 + \lambda_k)H_0\varphi'(y_0(t)))v_i\|_\alpha\end{aligned}$$

Since $\lambda_1 = 0 \in \text{eig}(\Gamma)$ we see that the index of non-singularity of the individual oscillators cannot be improved. For the remaining modes we have the following result

Lemma 1. Suppose for some $\hat{c}$ we have

$$\|(I - (1 + \lambda_k)H_0\varphi'(y_0(t)))v_i\|_\alpha \geq \hat{c}\|v_i\|_\alpha, \quad \forall v_i \in \mathbf{L}_{2\alpha} \tag{16}$$

for each $\lambda_k \in \text{eig}(\Gamma) \setminus \{0\}$. Then for all $v \perp S = \{\mathbf{1} \otimes v_i : v_i \in \mathbf{L}_{2\alpha}[0, \infty)\}$ we have

$$\|(I - L_0)v\|_\alpha \geq \hat{c}\|v\|_\alpha$$

Proof. Let $L_i = H_0\varphi'(y_0(t))$. By homogeneity it is no restriction to assume v has unit length. Each unit length vector $v \perp S$ can be represented as $v = \sum_{k=2}^N \alpha_k v_k \otimes v_{i,k}$ where $v_k \perp \mathbf{1}$ is a unit length eigenvector corresponding to $\lambda_k \in \text{eig}(\Gamma) \setminus \{0\}$, $\sum_{k=2}^N \alpha_k^2 = 1$ and $v_{i,k} \in \mathbf{L}_{2\alpha}[0, \infty)$ has $\|v_{i,k}\|_\alpha = 1$. By using that the eigenvectors v_k are orthogonal we get

$$\begin{aligned}\|(I-L_0)v\|_\alpha^2 &= \|\sum_{k=2}^{N}\alpha_k v_k \otimes (I-(1+\lambda_k)L_i)v_{i,k}\|_\alpha^2 \\ &= \sum_{k=2}^{N}\alpha_k^2 \|v_k \otimes (I-(1+\lambda_k)L_i)v_{i,k}\|_\alpha^2 \\ &\geq \inf_{\substack{\lambda_k \in \operatorname{eig}(\Gamma)\setminus\{0\} \\ \|v_{i,k}\|_\alpha = 1}} \|(I-(1+\lambda_k)L_i)v_{i,k}\|_\alpha^2 \\ &\geq \hat{c}^2 = \hat{c}^2\|v\|_\alpha^2\end{aligned}$$

where we used that $\|v_k \otimes w\|_\alpha = \|w\|_\alpha$. □

For the small gain criterion in (ii) of Theorem 2 we use

$$L_\theta - L_0 = (I+\Gamma)\operatorname{diag}\,(\Delta L_1(z_{1,\theta},\theta),\ldots,\Delta L_N(z_{N,\theta},\theta)),$$

where

$$\Delta L_k(z_{k,\theta},\theta) = H_0(s/T_{k,\theta})\varphi'(y_{k,\theta},\theta) - H_0(s)\varphi'(y_0) + \Delta_k(s/T_{k,\theta})\varphi'(y_{k,\theta})$$

It is easy to see in the case when all $z_{k,\theta} = z_\theta$ and $\Delta_k = \Delta$ are equal, then $v = \frac{1}{\sqrt{N}}\mathbf{1} \otimes v_i$ gives $(\Delta L(z_\theta,\theta) = \Delta L_k(z_\theta,\theta))$

$$\begin{aligned}\|(L_\theta - L_0)v\|_\alpha &= \|\frac{1}{\sqrt{N}}\mathbf{1} \otimes \Delta L(z_\theta,\theta)v_i\|_\alpha \\ &= \|\Delta L(z_\theta,\theta)v_i\|_\alpha\end{aligned}$$

This shows that the small gain condition is no better than for the individual oscillators. Moreover, if $|I+\Gamma| \leq 1$, i.e. if the eigenvalues are $-2 \leq \lambda(\Gamma) \leq 0$, then the worst case norm for the network is no worse than the worst case norm for a single oscillator.

(iii) We first consider the computation of the induced norm of the right inverse $F_z'(z_0,0)^\dagger$. One possible realization of $F_z'(z_0,0)^\dagger : w \mapsto (v,\delta T)$ is (see [4])

$$\dot{x} =(I\otimes A + (I+\Gamma)\otimes B\varphi'(y_0)C)x + I\otimes B\varphi'(y_0)w + Kx(0),\ x(1) = x(0) \tag{17}$$

$$(v,\delta T) =(w + ((I+\Gamma)\otimes C)x, Kx(0))$$

where $K = (\mathbf{1} \otimes \dot{x}_0(0))^T$. Any bound $\gamma > \|F_z(z_0, 0)^\dagger\|$ satisfies

$$|Kx(0)|^2 + \int_0^1 (|w + ((I + \Gamma) \otimes C)x|^2 - \gamma^2|w|^2)dt \leq 0 \tag{18}$$

for all $w \in \mathbf{L}_2(1)$ subject to the dynamics in (17). Let us consider the choice $w = v_k \otimes w_i$ and $x(0) = v_k \otimes x_i(0)$, where v_k is a unit length eigenvector corresponding to the eigenvalue $\lambda_k \in \text{eig}(\Gamma)$. Then (18) reduces to ($K_i = \dot{x}_0^T(0)$)

$$\begin{aligned}|v_k \otimes K_i x_i(0)|^2 + \int_0^1 (|v_k \otimes (w_i + (1 + \lambda_k)Cx_i)|^2 - \gamma^2|v_k \otimes w_i|^2)dt \\ = |K_i x_i(0)|^2 + \int_0^1 (|w_i + (1 + \lambda_k)Cx_i|^2 - \gamma^2|w_i|^2)dt \leq 0\end{aligned}$$

for all $w_i \in \mathbf{L}_2(1)$ subject to

$$\dot{x}_i = (A + (1 + \lambda_k)B\varphi'(y_0)C)x_i + B\varphi'(y_0)w_i + K_i x_i(0), \;\; x_i(1) = x_i(0) \tag{19}$$

In particular, the case $(v_k, \lambda_k) = (\mathbf{1}, 0)$ shows that the optimization problem for an individual oscillator is recovered and thus the same norm bound is achieved. The network should be designed to obtain as good bounds as possible for the remaining eigenmodes of the coupling matrix. For this we use the following lemma

Lemma 2. Suppose that for some $\hat{\gamma}$ and for all $\lambda_k \in \text{eig}(\Gamma) \setminus \{0\}$

$$|K_i x_i(0)|^2 + \int_0^1 (|w_i + (1 + \lambda_k)Cx_i|^2 - \hat{\gamma}^2|w_i|^2)dt \leq 0 \tag{20}$$

for any solution of (19). Then condition (18) is satisfied with $\gamma = \hat{\gamma}$ for all $w \perp S = \{\mathbf{1} \otimes w_i : w_i \in \mathbf{L}_2(1)\}$.

Proof. By homogeneity it is no restriction to assume w has unit length. A unit length vector $w \perp S$ can be decomposed as $w = \sum_{k=2}^N \alpha_k v_k \otimes w_{i,k}$ where $v_k \perp \mathbf{1}$ is a unit length eigenvector corresponding to $\lambda_k \in \text{eig}(\Gamma) \setminus \{0\}$, $w_{i,k} \in \mathbf{L}_2(1)$ with $\|w_{i,k}\| = 1$, and $\sum_{k=2}^N \alpha_k^2 = 1$. This input can be written $w = \sum_{k=2}^N \alpha_k w_k$ where $w_k = v_k \otimes w_{i,k}$ and it results in a state vector $x = \sum_{k=2}^N \alpha_k v_k \otimes x_{i,k}$, where $x_{i,k}$ satisfies (19) with λ_k and input $w_{i,k}$ (note we have a unique solution of (19) due to the condition on the characteristic multipliers (i)). If we define

$$J(w) = |Kx(0)|^2 + \int_0^1 |w + ((I+\Gamma)\otimes C)x|^2 dt$$

then

$$\begin{aligned} J(w) &= J(\sum_{k=2}^{N} \alpha_k w_k) = \sum_{k=2}^{N} \alpha_k^2 J(w_k) \\ &\leq \sup_{\substack{\lambda_k \in \operatorname{eig}(\Gamma)\setminus\{0\} \\ \|w_{i,k}\|=1}} J(w_k) \leq \hat{\gamma}^2 = \hat{\gamma}^2 \|w\|^2 \end{aligned}$$

This proves the lemma. □

For the small gain condition (ii) in Theorem 2 we need the worst case norm $\sup_{z\in\mathcal{Z},|\theta|\leq\bar{\theta}} \|F_z'(z,\theta) - F_z'(z_0,0)\|$. It is not hard to see that the bound for the network is no worse than for the individual oscillator if $-2 \leq \operatorname{eig}(\Gamma) \leq 0$. This same statement also holds for condition (iv) in Theorem 2.

To summarize the discussion we have seen that the worst case robustness of a symmetric oscillator network can be no better than the worst case robustness of the individual oscillators. The network design problem is to make the modes corresponding to unsynchronized states as robust as possible with as fast rate of convergence as possible. We separate the design procedure into two steps

1. For desired $\hat{c}$ and $\hat{\gamma}$ find a set Λ such that (16), (20) and the condition on the characteristic multipliers (i) hold for all $\lambda \in \Lambda$.

2. Determine the coupling matrix Γ such that $-2I \leq \Gamma \leq 0$ and $\operatorname{eig}(\Gamma)\setminus\{0\} \in \Lambda$.

We illustrate on the simpler problem of designing a network topology with given rate of synchronization.

Design of Networks

From Proposition 1 we see that the exponential decay rate is determined by the characteristic multipliers corresponding to the matrices $A_k(t) = A_{cl}(t) +$

$\lambda_k B\varphi'(y_0(t),0)C$, where $A_{cl}(t) = A + B\varphi'_y(y_0(t),0)C$ and $\lambda_k \in \sigma(\Gamma)$. Since, $\lambda_1 = 0 \in \sigma(\Gamma)$ it follows that n of the characteristic multipliers, including the one at 1, are identical to the characteristic multipliers of the individual nominal oscillators. If the network is designed such that the largest characteristic multiplier of the remaining system matrices is small then the network synchronization to the nominal solution is fast. We provide a formal definition

Definition 4. The synchronization rate for the network (11) is defined as

$$r = \max_{\text{eig}(\Gamma)\ni\lambda\neq 0} \max_{i=1,\dots,n} |\rho_i(A(\lambda))|$$

where $A(\lambda) = A + B(\varphi'_y(y_0(t),0) + \lambda\varphi'_{\bar{y}}(y_0(t),0))C$.

We want to design the network such that $r << 1$. The interpretation of the definition is given in the next proposition

Proposition 3. Consider the nominal variational system

$$\dot{x} = (I \otimes A_{cl}(t) + \Gamma \otimes M(t))x \tag{21}$$

where $A_{cl}(t) = A + B\varphi'_y(y_0(t),0)C$ and $M = B\varphi'_{\bar{y}}(y_0(t),0)C$ and $\Gamma = \Gamma^T$ has a simple eigenvalue at zero with eigenvector $\mathbf{1}$. Define the synchronized solution and the synchronized initial state as

$$\begin{aligned} S &= \{\mathbf{1} \otimes x_i(t) : \dot{x}_i(t) = A_{cl}(t)x_i(t)\} \\ S_0 &= \{\mathbf{1} \otimes x_{i0} : x_{i0} \in \mathbf{R}^n\} \end{aligned}$$

If the network has synchronization rate r then there exists $c > 0$ such that the solution of (21) satisfies

$$\text{dist}(x(t), S) \leq ce^{-\alpha t}|x_0|$$

for all $x_0 \in S_0^\perp$ and any $0 < \alpha < -\ln r$.

Proof. An arbitrary $x_0 \in S_0^\perp$ can be decomposed as

$$x_0 = \sum_{k=2}^{N}\sum_{l=1}^{n} \alpha_{k,l} v_k \otimes x_{k,l}(0)$$

where $v_k \perp \mathbf{1}$ is a unit length eigenvector corresponding to $\lambda_k \in \text{eig}(\Gamma) \setminus \{0\}$ and $x_{k,l}(0)$ is a unit length eigenvector corresponding to the characteristic multiplier $\rho_l(A_k)$, where A_k is defined in (13). The solution decomposes analogously

$$\begin{aligned}\dot{x}(t) &= \sum_{k=2}^{N}\sum_{l=1}^{n} \alpha_{k,l} v_k \otimes \dot{x}_{k,l}(t) \\ &= \sum_{k=2}^{N}\sum_{l=1}^{n} \alpha_{k,l} v_k \otimes A_k(t) x_{k,l}(t) \perp S\end{aligned}$$

This implies (($\Phi_k(t,0)$ is the transition matrix of $A_k(t)$)

$$\begin{aligned}\operatorname{dist}(x(t),S)^2 &= |x(t)|^2 \\ &= |\sum_{k=2}^{N}\sum_{l=1}^{n} \alpha_{k,l} v_k \otimes \Phi_k(t,0) x_{k,l}(0)|^2 \\ &= |\sum_{k=2}^{N} v_k \otimes \Phi_k(t,0) \sum_{l=1}^{n} \alpha_{k,l} x_{k,l}(0)|^2 \\ &= \sum_{k=2}^{N} |\Phi_k(t,0) \sum_{l=1}^{n} \alpha_{k,l} x_{k,l}(0)|^2 \\ &\le \sum_{k=2}^{N} c_k^2 e^{-2\alpha_k t} |\sum_{l=1}^{n} \alpha_{k,l} x_{k,l}(0)|^2 \\ &\le c e^{-2\alpha t} \sum_{k=2}^{N} |\sum_{l=1}^{n} \alpha_{k,l} x_{k,l}(0)|^2 = c e^{-2\alpha t} |x_0|^2\end{aligned}$$

where we used $|\Phi_k(t,0)| \le c_k e^{-\alpha_k t}$ for some $c_k > 0$, $c = \max_k(c_k^2)$, $0 < \alpha_k < -\ln(\max_i |\rho_i(A_k)|)$. At two places we use that the v_k are orthonormal. □

Attention will be restricted to the special case when the network has the further symmetry condition

$$\gamma_{k,l} = \gamma_{|k-l|} \quad \text{and} \quad \gamma_{N-k} = \gamma_k \tag{22}$$

In this case the coupling matrix is the circulant matrix

$$\Gamma = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_1 \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_2 \\ \gamma_2 & \gamma_1 & \gamma_0 & \cdots & \gamma_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_0 \end{bmatrix} \tag{23}$$

where $\gamma_0 = -\sum_{i=1}^{N-1} \gamma_k$. The eigenvalues of a circulant matrix are the Fourier transform of the first row. Exploiting that $\gamma_{N-k} = \gamma_k$ gives the real valued eigenvalues

$$\lambda_k = \sum_{l=0}^{N-1} e^{-2\pi kl/N} \gamma_l = 2 \sum_{l=1}^{\lfloor N/2 \rfloor} \gamma_l (\cos(2\pi kl/N) - 1) \tag{24}$$

We can design the network interconnection for given synchronization rate as follows

1. Find $\Lambda = \{\lambda \in \mathbf{R} : \max_{i=1,\ldots,n} |\rho_i(A(\lambda))| < r\}$ where $0 < r < 1$ is the desired synchronization rate.

2. Find an optimal network topology and interconnection strengths such that $\lambda \in \Lambda$ for all $\lambda \in \text{eig}(\Gamma) \setminus \{0\}$. There are several possible optimization problems. One example is the integer program

$$\min -\gamma^{-1} + \sum_{k=1}^{\lfloor N/2 \rfloor} d_k z_k \quad \text{subj. to}$$
$$\begin{cases} \sum_{l=1}^{\lfloor N/2 \rfloor} 2 z_k (\cos(2\pi kl/N) - 1) \in \gamma^{-1}\Lambda, \\ \gamma^{-1} \geq \epsilon; \; z_k \in \{0, 1\}, \forall k = 1, \ldots, \lfloor N/2 \rfloor \end{cases}$$

where ϵ is a positive number. Here the binary variables determine the topology of the network. The cost variables d_k can, for example, be chosen as an increasing sequence in order to achieve localized coupling, i.e. each oscillator is only interconnected to a few neighbors. To obtain a linear program we maximize the inverse of the interconnection strength, which must be chosen strictly positive.

Another example is the integer program of the form

$$\min \sum_{k=1}^{\lfloor N/2 \rfloor} (c_k\gamma_k + d_k z_k) \quad \text{subj. to}$$
$$\begin{cases} \sum_{l=1}^{\lfloor N/2 \rfloor} 2\gamma_l(\cos(2\pi kl/N) - 1) \in \Lambda, \\ 0 \le \gamma_k \le z_k M;\; z_k \in \{0,1\},\; \forall k = 1, \dots, \lfloor N/2 \rfloor \end{cases}$$

Once again the binary variables determine the topology of the network but here the interconnection strengths are optimized individually.

We apply this procedure to an example.

Example 3. In this example we will study a network of Van der Pol oscillators. Two different interconnection structures are considered.
1. A linear coupling on the form

$$\ddot{y}_k + m(y_k^2 - 1)\dot{y}_k + y_k = \theta\Delta_k y_k + \bar{y}_k \tag{25}$$

where $\bar{y}_k = \sum_{l \neq k} \gamma_{k,l}(y_l - y_k)$. This network can be represented as

$$\begin{aligned} \dot{x}_k(t) &= Ax_k(t) + B_1\phi(y_k(t), \bar{y}_k(t)) + \theta B_2(\Delta y_k)(t) \\ y_k(t) &= Cx_k(t) \end{aligned}$$

where A, B_2 and C are defined as in Example 1 while

$$\phi(y, \bar{y}) = \begin{bmatrix} -my^3/3 + (2+m)y \\ \bar{y} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $\bar{y}_k = \sum_{l \neq k} \gamma_{k,l}(y_l - y_k)$ under the symmetry assumption in (22).
2. A nonlinear coupling of the form

$$\ddot{y}_k + m((y_k + \bar{y}_k)^2 - 1)(\dot{y}_k + \dot{\bar{y}}_k) - 2\dot{\bar{y}}_k + y_k = \theta\Delta_k y_k$$

where $\bar{y}_k = \sum_{l \neq k} \gamma_{k,l}(y_l - y_k)$. Using the new states

$$\begin{aligned} x_{1,k} &= -\dot{y}_k - m((y_k + \bar{y}_k)^3/3 - y_k - \bar{y}_k) + 2\bar{y}_k \\ x_{2,k} &= y_k \end{aligned}$$

gives the system equation

$$\begin{aligned} \dot{x}_k &= Ax_k + B_1\varphi(y_k, \bar{y}_k) + \theta B_\Delta(\Delta y_k)(t) \\ y_k &= Cx_k \end{aligned}$$

N	r	$\frac{\lambda_2}{r}$	Λ	M	γ
10	0.6	0.6	$[-0.9, -0.2]$	2	0.18
10	0.2	1.7	$[-2.2, -0.25]$	2	0.23
10	0.034	10	$[-1.2, -0.5]$	3	0.27
10	0.0034	100	$[-1.0, -0.9]$	5	0.40
10	0.034	10	$[-1.2, -0.5]$	3	0.11
20	0.034	10	$[-1.2, -0.5]$	4	0.087
40	0.034	10	$[-1.2, -0.5]$	5	0.08
60	0.034	10	$[-1.2, -0.5]$	6	0.067
80	0.034	10	$[-1.2, -0.5]$	7	0.054
100	0.034	10	$[-1.2, -0.5]$	10	0.043
25	0.2	1.7	$[-2.2, -0.25]$	3	0.15
25	0.034	10	$[-1.2, -0.5]$	5	0.072
25	0.0034	100	$[-1.0, -0.9]$	24	0.019

where $\varphi(y, \bar{y}) = \phi(y + \bar{y})$ and $\phi(y) = -my^3/3 + (2+m)y$, A, B_1, B_Δ and C are defined as in Example 1 and $\bar{y}_k = \sum_{l \neq k} \gamma_{k,l}(y_l - y_k)$ under the symmetry assumption in (22).

For the dynamics of the Van der Pol oscillator we let $m = 0.2$. We consider a network with N oscillators. For the results in Table 3 we used $(d_1, d_2, d_3, d_4, d_5) = (1, 10, 100, 500, 1000)$, $d_k = 1000$ for $k > 5$ and M refers to the number of interconnections. The results were obtained using Xpress-MP on the NEOS Server.

Robustness to Perturbations of the Network

We have seen that the interconnection matrix Γ determines the stability and the robustness of the network. It is therefore important to design this matrix such that the network is robust to the failure of individual links and to the removal of nodes. We have the following result

Proposition 4. Suppose the bi-directional connection between node k and l is broken. The resulting interconnection matrix is

$$\hat{\Gamma} = \Gamma + \gamma_{k,l} e_{k,l} e_{k,l}^T$$

where $\gamma_{k,1} \geq 0$ and

$$e_{k,l} = \begin{cases} 1, & i = k \\ -1, & i = l \\ 0, & otherwise \end{cases}$$

$$e_{k,l}e_{k,l}^T = \begin{bmatrix} & \vdots & & \vdots & \\ \dots & 1 & \dots & -1 & \dots \\ & \vdots & & \vdots & \\ \dots & -1 & \dots & 1 & \dots \\ & \vdots & & \vdots & \end{bmatrix} \tag{26}$$

We have

$$\lambda_N(\hat{\Gamma}) = 0$$
$$\lambda_i(\Gamma) \leq \lambda_i(\hat{\Gamma}) \leq \min(\lambda_{i+1}(\Gamma), \lambda_i(\Gamma) + 2\gamma_{k,l})$$

The proof is a special case of the following general result

Proposition 5. Suppose the interconnection between the nodes $\{(k_1, l_1), \dots, (k_m, l_m)\}$ are lost. Then

$$\hat{\Gamma} = \Gamma + E, \quad E = \sum_{i=1}^{m} \gamma_{k_i,l_i} e_{k_i,l_i} e'_{k_i,l_i}$$

where $\gamma_{k,l} \geq 0$ and the e_{k_i,l_i} are defined as in (26). We have

$$\lambda_N(\Gamma + E) = 0 \tag{27}$$
$$\lambda_i(\Gamma) \leq \lambda_i(\Gamma + E) \leq \min_{i \leq k \leq N} (\lambda_{i+N-k}(\Gamma) + \lambda_k(E)) \tag{28}$$

In particular, if node k in the network fails then the resulting coupling matrix becomes

$$\hat{\Gamma} = \mathcal{T} \left(\Gamma + \sum_{i \in \mathcal{N}(k)} \gamma_{k,i} e_{k,i} e'_{k,i} \right) \mathcal{T}^T$$

where $\mathcal{N}(k)$ are the nodes to which k is connected and

$$\mathcal{T} = \begin{bmatrix} I_{k-1} & 0_{(k-1),1} & 0_{(k-1),(N-k)} \\ 0_{(N-k),(k-1)} & 0_{(N-k),1} & I_{N-k} \end{bmatrix}$$

The eigenvalues of the perturbed matrix are now related as

$$\lambda_{N-1}(\hat{\Gamma}) = 0$$
$$\lambda_i(\Gamma) \le \lambda_i(\hat{\Gamma}) \le \min_{i \le k \le N-1} (\lambda_{i+N-k}(\Gamma) + \lambda_k(E))$$

for $i = 1, \dots, N-2$.

Remark 1. From the above inequalities we see that

$$\lambda_i(\Gamma + E) \le \min(\lambda_{i+m}(\Gamma), \lambda_i(\Gamma) + \lambda_{\max}(E))$$

The first part follows by letting $k = N - m$ and using that $\lambda_{N-m}(E) = 0$ since E has at most rank m. The second inequality follows by using $k = N$.

Proof. By Weyl's theorem (Theorem 4.3.7 in [1]) we have

$$\lambda_{j+k-N}(\Gamma + E) \le \lambda_j(\Gamma) + \lambda_k(E)$$

for all $1 \le j, k \le N$, $j + k \ge N + 1$. Let $i = j + k - N$. Then the above can be written

$$\lambda_i(\Gamma) \le \lambda_i(\Gamma + E) \le \min_{i \le k \le N} (\lambda_{i+N-k}(\Gamma) + \lambda_k(E)) \tag{29}$$

The left inequality is trivial and sufficient for our purposes. By the structure of the matrices we have $\lambda_N(\Gamma+E) = 0$. This is a consequence of the Geršgorin theorem (Theorem 6.1.1 in [1]. Indeed, $\mathbf{1}$ is an eigenvector corresponding to the eignvalue 0. Since the spectrum of $\Gamma + E$ belongs to the discs (the first case with perturbation E is treated similarly)

$$\cup_{i=1}^{N} \{z \in \mathbf{C} : |z + \sum_{j \notin \mathcal{N}(k)} \gamma_{i,j}| \le \sum_{j \notin \mathcal{N}(k)} \gamma_{i,j}\}$$

where we use that all $\gamma_{i,j} \ge 0$ and $\mathcal{N}(k)$ denotes the neighbors of k that are removed. Clearly $\lambda_{\max}(\Gamma + E) = 0$, which proves the statement. □

We will next discuss the properties of large networks. We have seen in the example above that the number of interconnections in the network increase with the demand on improved synchronization rate. This property and certain robustness properties of the network have been discussed in several works before, see e.g. [8, 15]. Here we will try to review and bring some new interpretation of these issues. For symmetric couplings as in (23) all nodes in the network are connected when at least one interconnection strength γ_k is nonzero. To understand large connected networks we consider $\gamma = \begin{bmatrix}\gamma_1 & \gamma_2 & \dots\end{bmatrix} \in l_1(Z^+)$. The boundedness assumption reflects the physical constraint that only a finite total interconnection strength can be applied at each node. Hence, for large N the eigenvalues in (24) can be approximated by the sum

$$\lambda(\omega) = \sum_{l=1}^{\infty} (\cos(\omega l) - 1)\gamma_l$$

for $\omega \in [0, 2\pi]$. Note that $\lambda(0) = 0$ and since $\lambda(\cdot)$ is a continuous function it follows that there exists some interval near $\omega = 0$ where $|\lambda(\omega)| \leq \epsilon$ for any $\epsilon > 0$. This implies that in large networks the synchronization rate is near zero. Hence, large networks will from a dynamic point of view effectively be disconnected or put in other terms, the synchronization effect is only present in local clusters while far away nodes do not communicate.

By Parceval's theorem we have

$$\langle \hat{v}, \lambda \hat{v} \rangle_{\mathbf{L}_2[0,2\pi]} = \langle v, \Gamma v \rangle_{l_2(Z)} \tag{30}$$

where Γ is the infinite dimensional circulant matrix corresponding to $\gamma \in l_1(Z^+)$, $v \in l_2(Z)$. Due to the symmetry of Γ it is no restriction to assume $v_{-k} = v_k$ and therefore $\hat{v} \in \mathbf{L}_2[0, 2\pi]$, the corresponding discrete Fourier transform, can be computed as

$$\hat{v}(\omega) = \sum_{k=0}^{\infty} v_k \cos(\omega k)$$

Expression (30) can be viewed as the limit of the eigenvalue decomposition as the network size tends to infinity. This can be used to understand certain robustness properties of large networks. Indeed, the interlacing property of the eigenvalues in Proposition 5 together with the dense clustering of the eigenvalues for a large network imply that the network will be insensitive to the removal of interconnections and nodes. It has recently been argued that a special type of interconnection topologies, "small world networks", combine the robustness of the regular networks discussed here with the fast synchronization of large random networks, see [16].

4 Appendix. Proof of Proposition 2

Let

$$\hat{\theta} = \sup\{\bar{\theta} : \bar{\theta} \text{ is a robustness margin of (10)}\}.$$

For any individual oscillator let $F_k(z_{k,\theta}, \theta) = y_{k,\theta} - H_k(s/T_{k,\theta}, \theta)\varphi(y_{k,\theta}, \bar{y}_{k,\theta})$. By assumption we have $F_k(z_{k,\theta}, \theta) \equiv 0$ on $(-\hat{\theta}, \hat{\theta})$. Differentiation of this inequality gives

$$\frac{d}{dz}F_k(z_{k,\theta}, \theta)\frac{dz_{k,\theta}}{d\theta} = -\frac{d}{d\theta}F_k(z_{k,\theta}, \theta)$$

The non-uniqueness of $z_{k,\theta}$ due to time translation of the trajectory $y_{k,\theta}$ implies that $(\dot{y}_{k,\theta}, 0) \in \mathrm{Ker}\frac{d}{dz}F_k(z_{k,\theta}, \theta)$, see [4]. It is shown in [4] that $\dim \mathrm{Ker}\frac{d}{dz}F_k(z_{k,\theta}, \theta) = 1$ is a necessary and sufficient condition for the existence of a right inverse such that

$$\frac{dz_{k,\theta}}{d\theta} = -\left(\frac{d}{dz}F_k(z_{k,\theta}, \theta)\right)^{\dagger}\frac{d}{d\theta}F_{k,}(z_{k,\theta}, \theta)$$

It can also be shown that the only non-uniqueness of the derivative corresponds to time translation. On the contrary, if the right inverse of $\frac{d}{dz}F_k(z_{k,\theta}, \theta)$ ceases to exist at $\theta = \theta_0$ then the uniqueness property of the derivative fails at this point (the limit cycle may even cease to exist. We conclude that the right inverse must exist on $(-\hat{\theta}, \hat{\theta})$.

Now consider the network (12) in the case when $H_{k_1}(s, \theta) = H_{k_2}(s, \theta)$ for all $k_1, k_2 = 1, \dots, N$. Then

$$z_\theta = \left[\hat{z}_\theta^T, \dots, \hat{z}_\theta^T\right]^T \tag{31}$$

is a synchronized solution for $\theta \in (-\hat{\theta}, \hat{\theta})$, i.e. all oscillators have the same solution $z_{k,\theta} = \hat{z}_\theta$, $k = 1, \dots, N$. We will show that it cannot be continued any further by proving that any of the following three conditions hold true

(*i*) existence of a unique solution fails, i.e. a bounded right inverse of $F_z'(z_\theta, \theta)$ ceases to exist at either of the boundary points $\{-\hat{\theta}, \hat{\theta}\}$.

(*ii*) $z_\theta \notin \mathcal{Z}$ for any sufficiently small $\theta > \hat{\theta}$ or any sufficiently large $\theta < -\hat{\theta}$.

(*iii*) the exponential stability is lost, or equivalently $\mathrm{def}_\alpha(L(z_\theta, \theta)) = 1$ fails at either of the boundary points $\{-\hat{\theta}, \hat{\theta}\}$ (note that according to the stability definition, stability ceases to hold when the operator becomes singular).

By assumption, any of $(i)-(iii)$ must hold for the individual oscillators. We will use that

$$L(z_\theta,\theta) = (I+\Gamma)\otimes H_k(s/\hat{T}_\theta,\theta)\varphi_y'(\hat{y}_\theta,0)$$

and

$$F_z'(z_\theta,\theta) = \left[I-(I+\Gamma)\otimes L_{k,st}(\hat{z}_\theta,\theta) \quad \frac{1}{T_\theta}(I+\Gamma)\otimes(I-L_{k,st}(\hat{z}_\theta,\theta))(t\dot{\hat{y}}_\theta) \right]$$

From the structure of these operators we immediately see that if (i) or (iii) are true for the individual oscillator then the corresponding condition holds for the corresponding network operator. For example, consider (iii). Let $L_k(\hat{z}_\theta,\theta) = H_k(s/\hat{T}_\theta,\theta)\varphi_y'(\hat{y}_\theta,0)$. If $L_k(\hat{z}_\theta,\theta)v_k = v_k$, i.e. the operator is singular and the stability defect condition fails and so does local exponential stability. Since also $L(z_\theta,\theta)(\mathbf{1}\otimes v_k) = \mathbf{1}\otimes v_k$, the same conclusion holds for the network.

Finally, suppose (ii) holds true for the individual operator. Then

$$\lim_{\theta\nearrow\hat{\theta}} \|\hat{z}_\theta\|_{C(1)\times\mathbf{R}} = r_0$$
$$\lim_{\theta\nearrow\hat{\theta}} \frac{d}{d\theta}\|\hat{z}_\theta\|_{C(1)\times\mathbf{R}} > 0$$

or similarly for the other limit. Obviously, since (31) is the network solution on $(-\hat{\theta},\hat{\theta})$ the corresponding derivate limits also hold for the network solution.

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