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Trajectory planning for systems with a multiplicative stochastic uncertainty

ULF T. JÖNSSON†*, CLYDE MARTIN‡ and YISHAO ZHOU§

A trajectory planning problem for linear stochastic differential equations is considered in this paper. The aim is to control the system such that the expected value of the output interpolates given points at given times while the variance and an integral cost of the control effort are minimized. The solution is obtained by using the dynamic programming technique, which allows for several possible generalizations. The results of this paper can be used for control of systems with a multiplicative stochastic disturbance on the state vector and systems with a stochastic growth rate. This is frequently the case in investment problems, biomathematics and control theory.

1. Introduction

The theory of control theoretic splines is concerned with controlling the state or some measured quantity of the state to given values at discrete times. That is, given a system of the form

$$\dot{x} = f(x) + u_1 g_1(x) + \cdots + u_k g_k(x)$$

and

$$y = Cx$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$, we look for a system of controls that drive the output through or close to a prescribed sequence of set points

$$\{ y(t_i) = \alpha_i \colon i = 1, \ldots, N \}.$$

We call the generated output function an *interpolating spline* if it exactly interpolates these points, otherwise it is called a *smoothing spline*. The term *control theoretic spline* refers to the way the spline function is constructed, i.e. as the output of a dynamical system driven by a control function. The dynamics can either be a model of a system to be controlled or an auxiliary model that helps to define the form of the constructed spline function. Much of the literature surveyed below considers the case when the dynamics is linear and the spline function is generated as the solution of an optimal control problem. In this paper we consider a version of this problem where there is a multiplicative stochastic uncertainty in the dynamics. It is then no longer possible to obtain exact interpolation and instead we let the expected value interpolate the given point at the same time as the variance and the expected value of an integral quadratic cost along the trajectory are minimized. By minimizing the variance at the interpolation points we minimize the effect of uncertainty due to the diffusion term in the system equation.

Interpolating splines have been important since the 1960s as a tool in numerical analysis. A quick review of the literature shows a tremendous jump in the number of papers related to splines around 1970 which corresponds to the availability of computers for numerical calculations. Smoothing splines were first used in the 1960s but it wasn't until Grace Wahba did a systematic development that they become important in statistics. Wahba's book (Wahba 1990) is an important reference that is basic to understanding the myriad applications of smoothing splines as a tool in non-parametric statistics. The first appearance of control theoretic interpolating splines was an application to trajectory planning for aircraft in a seminal paper of Crouch and Jackson (1990). Wahba's development of smoothing splines was very much in the spirit of control theoretic smoothing splines but she was only interested in polynomial splines. To our knowledge the first work on general control theoretic smoothing splines was in Sun *et al.* (2000). Control theoretic splines have been applied to trajectory planning for aircraft (Crouch and Jackson 1990), to wildlife tracking (Egerstedt and Martin 2003), to trajectory planning for robots (Egerstedt and Martin 2000) and, of course, to many problems in curve estimation and to many different problems in statistics.

Interpolating splines are useful when the data is known to be exact and it is required for a curve to exactly pass through a specific set of points. In most applications exactness is not required and it suffices

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*Author for correspondence. e-mail: ulfj@math.kth.se
† Optimization and Systems Theory, Royal Institute of Technology, SE-100 44 Stockholm, Sweden.
‡ Department of Mathematics and Statistics, Texas Tech University, Lubbock, Texas, USA.
§ Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden.

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for the data to be close to the constructed curve. Smoothing splines were designed for exactly this purpose. In statistics the general object is to fit a curve to data in such a manner that the error between the data and the constructed curve has nice statistical properties, for example normally distributed. For control theoretic splines the object was and is slightly different. Here we want a trajectory that is close to the given data points but we seek a trade-off between exactness and cost. For example in aircraft applications, exactness often comes at the price of large accelerations and hence increases in fuel usage. In this paper we take a different approach. We place the uncertainty in the dynamics and require exactness in terms of expected value. So both this approach and the approach of smoothing splines have the effect of filtering the noisy data. For many problems the approach of this paper is the most natural.

While there is a great deal of literature on the problem of constructing splines using deterministic methods there is little if any literature on the construction of splines using stochastic dynamics. However, there are many problems ranging from biological to economic to trajectory planning where the dynamics can be naturally considered as stochastic. This is the motivation for writing this paper. In earlier work by Zhou and Martin and other collaborators the problem of deterministic splines was attacked by obtaining explicit solutions to the output of the system and then optimizing the cost over the control; see for example the techniques in Martin *et al.* (2001). In this paper a more powerful technique is used for the optimization–dynamic programming. This is a necessary step to finding the solution and allows one to use standard techniques of solving the problem locally and showing the explicit dependence of the solution on the Riccati equation. It has the additional advantage in that it directly leads to feedback solutions of the corresponding control problem. We also solve the problem using inequality constraints on the output and we present a complete example of using the method for solving the trajectory planning problem for a mobile robot. Here the combined effect of surface irregularities, friction, and other disturbances is modeled as a multiplicative stochastic disturbance. Further examples and several possible generalizations are discussed in the concluding remarks.

2. Stochastic trajectory planning

We consider a trajectory planning problem where a stochastic disturbance enters the differential equation. As our basic problem we consider

$$J_0(x_0) = \min_{u\in\mathcal{M}(t_0,t_N)} \sum_{k=0}^{N-1} w_{k+1}\mathrm{Var}\{CX(t_{k+1})\} + E^{t_0,x_0}\left\{\int_0^{t_N} |u|^2 \mathrm{d}t\right\}$$
$$\text{subject to } \begin{cases} \mathrm{d}X = (AX+Bu)\,\mathrm{d}t + G(X)\,\mathrm{d}Z, \quad X(t_0)=x_0 \\ E^{t_0,x_0}\{CX(t_{k+1})\} = \alpha_{k+1}, \\ \qquad k=0,\dots,N-1 \end{cases} \tag{1}$$

where $w_k \geq 0$ and (A, B) is a controllable pair. The last term in the stochastic differential equation corresponds to a multiplicative noise on the state vector defined by an m-dimensional Brownian motion Z_t. In other words, Z_t is a process with zero mean and independent increments, i.e. $E\{\mathrm{d}Z_t\,\mathrm{d}Z_s^{\mathrm{T}}\} = \delta(t-s)I\mathrm{d}t$, where δ denotes the dirac function. We assume that $G(x)$ is a linear function on the form

$$G(x) = \sum_{i=1}^{n} x_i G_i$$

where $G_k \in \mathbb{R}^{n\times m}$. In (1), E^{t_0,x_0} is the expectation operator given that $X(t_0)=x_0$ and $\mathrm{Var}\{CX(t_k)\} = E^{t_0,x_0}\{(CX(t_k) - E^{t_0,x_0}\{CX(t_k)\})^2\}$.

We consider optimization over all Markov controls, i.e. feedback controls on the form $u(t,\omega) = \mu(t, X(t,\omega))$. We let $\mathcal{M}(t_0,t_N)$ denote the set of Markov controls on an interval (t_0,t_N). It can be shown that our optimal solution is also optimal over all $\mathcal{F}_t$ adapted processes, where $\mathcal{F}_t$ denotes the σ algebra generated by Z_s for $s \leq t$ (see, e.g. Øksendal 1998).

It turns out that there will be linear and constant terms in the value function due to the variance term in the objective function. It is therefore no essential addition in complexity to consider a more general path planning problem, where we allow the dynamics to be time-varying and different from stage to stage. We also consider integral costs that penalize the state vector. This gives the following generalization of (1)

$$J_0(x_0) = \min_{u\in\mathcal{M}(t_0,t_N)} E^{t_0,x_0}\left\{\sum_{k=0}^{N-1}\left(w_{k+1}|C_{1,k+1}X(t_{k+1}) - \beta_{k+1}|^2 + \int_{t_k}^{t_{k+1}} \sigma_k(t,X,u)\,\mathrm{d}t\right)\right\}$$
$$\text{subject to } \begin{cases} \mathrm{d}X = (A_k(t)X + B_k(t)u)\,\mathrm{d}t + G_k(t,X)\,\mathrm{d}Z, \quad t\in[t_k,t_{k+1}], \\ \qquad X(t_0)=x_0 \\ E^{t_0,x_0}\{C_{2,k+1}X(t_{k+1})\} = \alpha_{k+1}, \\ \qquad k=0,\dots,N-1 \end{cases} \tag{2}$$

where

$$\sigma_k(t,x,u) = x^{\mathrm{T}}Q_k(t)x + 2x^{\mathrm{T}}S_k(t)u + u^{\mathrm{T}}R_k(t)u + 2q_k(t)^{\mathrm{T}}x + 2r_k(t)^{\mathrm{T}}u + \varrho_k(t)$$

and

$$G_k(t,x) = \sum_{i=1}^{n} x_i G_{k,i}(t)$$

and where everything else is defined in a similar way as in (1). Here $A_k, B_k, Q_k, \ldots, \rho_k$ and $G_{k,i}$ are piecewise continuous functions of time and all pairs $(A_k(t), B_k(t))$ are assumed to be completely controllable and the cost is strictly convex, which is the case if for all $t \in [t_k, t_{k+1}]$, $k = 0, \ldots, N-1$, we have

$$R_k(t) > 0 \quad \text{and} \quad \begin{bmatrix} Q_k(t) & S_k(t) \\ S_k(t)^{\mathrm{T}} & R_k(t) \end{bmatrix} \geq 0. \tag{3}$$

Note that if $C_1 = C_2 = C$ and $\alpha_k = \beta_k$ then $E^{t_0,x_0}\{|CX(t_k) - \beta_k|^2\} = \mathrm{Var}\{CX(t_k)\}$.

Let us define the cost-to-go functions

$$J_k(x) = \min_{u\in\mathcal{M}(t_k,t_N)} E^{t_k,x}\left\{\sum_{i=k}^{N-1}\left(w_{i+1}|C_{1,i+1}X(t_{i+1}) - \beta_{i+1}|^2 + \int_{t_i}^{t_{i+1}} \sigma_i(t,X,u)\,\mathrm{d}t\right)\right\}$$

$$\text{subject to} \begin{cases} \mathrm{d}X = (A_i(t)X + B_i(t)u)\,\mathrm{d}t + G_i(t,X)\,\mathrm{d}Z, \\ t\in[t_i,t_{i+1}], \qquad X(t_k) = x \\ E^{t_0,x_0}\{C_{2,i+1}X(t_{i+1})\} = \alpha_{i+1}, \\ \qquad\qquad i = k,\ldots,N-1 \end{cases} \tag{4}$$

$$J_N(x) = 0.$$

Due to the Markov property of the stochastic differential equation we can use dynamic programming to solve (1). We next state two propositions and then the main result that solves (1). The first states the dynamic programming recursion.

Proposition 1: *The optimal cost satisfies the following dynamic programming equation*

$$J_k(x) = \min_{u\in\mathcal{M}(t_k,t_{k+1})} E^{t_k,x}\left\{\int_{t_k}^{t_{k+1}} \sigma_k(t,X,u)\,\mathrm{d}t + w_{k+1}|C_{1,k+1}X(t_{k+1}) - \beta_{k+1}|^2 + J_{k+1}(X(t_{k+1}))\right\}$$

$$\text{subject to} \begin{cases} \mathrm{d}X = (A_k(t)X + B_k(t)u)\,\mathrm{d}t \\ \qquad + G_k(t,X)\,\mathrm{d}Z, \quad X(t_k) = x \\ E^{t_k,x}\{C_{2,k+1}X(t_{k+1})\} = \alpha_{k+1} \end{cases}$$

$$J_N(x) = 0.$$

Proof: The proof is based on a standard dynamic programming argument and is given in the appendix for completeness. □

The next proposition states the solution to the stochastic optimal control problem in (5) below. It shows that $J_k(x)$ is a quadratic function which is instrumental in solving the dynamic programming iteration. Let

$$V(x_0,\alpha,t_0,t_f) = \min_{u\in\mathcal{M}(t_0,t_f)} E^{t_0,x_0}\left\{\int_{t_0}^{t_f} \sigma(t,X(t),u(t))\,\mathrm{d}t + X(t_f)^{\mathrm{T}}Q_0X(t_f) + 2q_0^{\mathrm{T}}X(t_f) + \varrho_0\right\}$$

$$\text{subject to} \begin{cases} \mathrm{d}X = (A(t)X + B(t)u)\,\mathrm{d}t + G(t,X)\,\mathrm{d}Z, \\ \qquad X(t_0) = x_0 \\ E^{t_0,x_0}\{CX(t_f)\} = \alpha \end{cases} \tag{5}$$

where

$$\sigma(t,x,u) = x^{\mathrm{T}}Q(t)x + 2x^{\mathrm{T}}S(t)u + u^{\mathrm{T}}R(t)u + 2q(t)^{\mathrm{T}}x + 2r(t)^{\mathrm{T}}u + \varrho(t)$$

and where Q, R and S satisfy the conditions in (3).

Proposition 2: *We have*

$$V(x_0,\alpha,t_0,t_f) = x_0^{\mathrm{T}}P(t_0)x_0 + 2p(t_0)^{\mathrm{T}}x_0 + \rho(t_0) + (N(t_0)^{\mathrm{T}}x_0 + m(t_0))^{\mathrm{T}}W(t_0)^{-1}(N(t_0)^{\mathrm{T}}x_0 + m(t_0))$$

where

$$\begin{aligned}
&\dot{P} + A^{\mathrm{T}}P + PA + Q + \Pi(P) \\
&\quad = (PB+S)R^{-1}(PB+S)^{\mathrm{T}}, \quad P(t_f) = Q_0 \\
&\dot{N} + (A - BR^{-1}(PB+S)^{\mathrm{T}})^{\mathrm{T}}N = 0, \quad N(t_f) = C^{\mathrm{T}} \\
&\dot{p} + (A - BR^{-1}(PB+S)^{\mathrm{T}})^{\mathrm{T}}p + q \\
&\quad = (PB+S)R^{-1}r, \quad p(t_f) = q_0 \\
&\dot{W} + N^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}N = 0, \quad W(t_f) = 0 \\
&\dot{m} = N^{\mathrm{T}}BR^{-1}(B^{\mathrm{T}}p + r), \quad m(t_f) = -\alpha \\
&\dot{\rho} + \varrho = (r + B^{\mathrm{T}}p)^{\mathrm{T}}R^{-1}(r + B^{\mathrm{T}}p), \quad \rho(t_f) = \varrho_0
\end{aligned} \tag{6}$$

and where $\Pi(P)$ is a linear matrix function with elements $\Pi(P)_{k,l} = \frac{1}{2}\mathrm{tr}(G_k^{\mathrm{T}}PG_l)$. The optimal control is

$$u^* = -R^{-1}(PB+S)^{\mathrm{T}}X - R^{-1}B^{\mathrm{T}}N\nu - R^{-1}(B^{\mathrm{T}}p + r)$$

with $\nu = -W(t_0)^{-1}(N(t_0)^{\mathrm{T}}x_0 + m(t_0))$.

Proof: The proof is done by Lagrangian relaxation of the linear constraint. See the appendix for the complete details. □

Remark 1: If Q, R and S satisfy condition (3), then the linearly perturbed Riccati equation in (6) has an absolutely continuous unique positive semidefinite solution (Wonham 1968). All other differential equations in (6) are linear with bounded piecewise continuous coefficients, which ensure existence of unique absolutely continuous solutions. Note also that we have $W(t) > 0$ for $t \in [t_0, t_f)$ by the complete controllability of the pair $(A(t), B(t))$.

Remark 2: If the stochastic term in (1) and (2) is removed then we obtain a deterministic trajectory planning problem. The solution to the deterministic problem is obtained by omitting the term $\Pi(P)$ in the Riccati equation above.

In order to obtain a compact notation we introduce

$$\widehat{Q}_0 = \begin{bmatrix} Q_0 & q_0 \\ q_0^{\mathrm{T}} & \varrho_0 \end{bmatrix}, \quad \widehat{Q} = \begin{bmatrix} Q & q \\ q^{\mathrm{T}} & \varrho \end{bmatrix}, \quad \widehat{S} = \begin{bmatrix} S \\ r^{\mathrm{T}} \end{bmatrix}$$
$$\widehat{P} = \begin{bmatrix} P & p \\ p^{\mathrm{T}} & \rho \end{bmatrix}, \quad \widehat{N} = \begin{bmatrix} N \\ m^{\mathrm{T}} \end{bmatrix}, \quad \widehat{W} = W, \quad \widehat{R} = R$$
$$\widehat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \widehat{C}(\alpha) = [C \quad -\alpha]$$
$$\hat{G}(t,x) = \begin{bmatrix} G(t,x) \\ 0 \end{bmatrix}, \quad \hat{\Pi}(\widehat{P}) = \begin{bmatrix} \Pi(P) & 0 \\ 0 & 0 \end{bmatrix}. \tag{7}$$

If we finally let $\widehat{X} = [X^{\mathrm{T}} \ 1]^{\mathrm{T}}$ and $\widehat{x}_0 = [x_0^{\mathrm{T}} \ 1]^{\mathrm{T}}$ then the optimization problem (5) can be written

$$V(x_0, \alpha, t_0, t_f) = \min_{u \in \mathcal{M}(t_0, t_f)} E^{t_0, x_0}\left\{ \int_{t_0}^{t_f} [\widehat{X}^{\mathrm{T}} \widehat{Q} \widehat{X} + 2\widehat{X}^{\mathrm{T}} \widehat{S} u + u^{\mathrm{T}} \widehat{R} u]\,\mathrm{d}t + \widehat{X}(t_f)^{\mathrm{T}} \widehat{Q}_0 \widehat{X}(t_f) \right\}$$
$$\text{subject to} \quad \begin{cases} \mathrm{d}\widehat{X} = (\widehat{A}(t)\widehat{X} + \widehat{B}(t)u)\,\mathrm{d}t + \widehat{G}(t,X)\,\mathrm{d}Z, \\ \widehat{X}(t_0) = \widehat{x}_0 \\ E^{t_0, x_0}\{\widehat{C}(\alpha)\widehat{X}(t_f)\} = 0 \end{cases} \tag{8}$$

and the optimal solution can be written

$$V(x_0, \alpha, t_0, t_f) = \widehat{x}_0^{\mathrm{T}} \Big[\widehat{P}(t_0) + \widehat{N}(t_0) \widehat{W}(t_0)^{-1} \widehat{N}(t_0)^{\mathrm{T}} \Big] \widehat{x}_0$$

where

$$\dot{\widehat{P}} + \widehat{A}^{\mathrm{T}} \widehat{P} + \widehat{P} \widehat{A} + \widehat{Q} + \hat{\Pi}(\widehat{P}) = (\widehat{P}\widehat{B} + \widehat{S})\widehat{R}^{-1}(\widehat{P}\widehat{B} + \widehat{S})^{\mathrm{T}}, \quad \widehat{P}(t_f) = \widehat{Q}_0$$
$$\dot{\widehat{N}} + (\widehat{A} - \widehat{B}\widehat{R}^{-1}(\widehat{P}\widehat{B} + \widehat{S})^{\mathrm{T}})^{\mathrm{T}} \widehat{N} = 0, \quad \widehat{N}(t_f) = \widehat{C}(\alpha)^{\mathrm{T}}$$
$$\dot{\widehat{W}} + \widehat{N}^{\mathrm{T}} \widehat{B} \widehat{R}^{-1} \widehat{B}^{\mathrm{T}} \widehat{N} = 0, \quad \widehat{W}(t_f) = 0.$$

The optimal control is

$$u^* = -\widehat{R}^{-1}(\widehat{P}\widehat{B} + \widehat{S})^{\mathrm{T}} \widehat{x} - \widehat{R}^{-1} \widehat{B}^{\mathrm{T}} \widehat{N} \widehat{W}(t_0)^{-1} \widehat{N}(t_0)^{\mathrm{T}} \widehat{x}_0.$$

There is an equivalent feedback form given as $u^* = -\widehat{R}^{-1}((\widehat{P} + \widehat{N}\,\widehat{W}^{-1}\widehat{N}^{\mathrm{T}})\widehat{B} + \widehat{S})^{\mathrm{T}} \widehat{X}$.

We can now state the solution of the general stochastic trajectory planning problem in (2).

Proposition 3: *Consider the optimal control problem in* (2), *where the condition* (3) *holds. The optimal Markov control in each time interval* $[t_k, t_{k+1}]$ *is (all variables are defined analogously with* (7)*)*

$$u^*(t) = -\widehat{R}_k(t)^{-1}(\widehat{P}_k(t)\widehat{B}_k(t) + \widehat{S}_k(t))^{\mathrm{T}} \widehat{X}(t) - \widehat{R}_k(t)^{-1} \widehat{B}_k(t)^{\mathrm{T}} \widehat{N}_k(t) \widehat{W}_k(t_k)^{-1} \widehat{N}_k(t_k)^{\mathrm{T}} \widehat{X}(t_k)$$

where for $k = N-1, \ldots, 0$

$$\dot{\widehat{P}}_k + \widehat{A}_k^{\mathrm{T}} \widehat{P}_k + \widehat{P}_k \widehat{A}_k + \widehat{Q}_k + \hat{\Pi}_k(\widehat{P}_k) = (\widehat{P}_k \widehat{B}_k + \widehat{S}_k)\widehat{R}_k^{-1}(\widehat{P}_k \widehat{B}_k + \widehat{S}_k)^{\mathrm{T}}$$
$$\dot{\widehat{N}}_k + (\widehat{A}_k - \widehat{B}_k \widehat{R}_k^{-1}(\widehat{P}_k \widehat{B}_k + \widehat{S}_k)^{\mathrm{T}})^{\mathrm{T}} \widehat{N}_k = 0,$$
$$\widehat{N}_k(t_{k+1}) = \widehat{C}_{2,k+1}(\alpha_{k+1})^{\mathrm{T}}$$
$$\dot{\widehat{W}}_k + \widehat{N}_k^{\mathrm{T}} \widehat{B}_k \widehat{R}_k^{-1} \widehat{B}_k^{\mathrm{T}} \widehat{N}_k = 0, \quad \widehat{W}_k(t_{k+1}) = 0$$

and where $\widehat{P}_{N-1}(t_N) = w_N \widehat{C}_{1,N}(\beta_N)^{\mathrm{T}} \widehat{C}_{1,N}(\beta_N)$ *and for* $k = N-2, N-3, \ldots, 0$

$$\widehat{P}_k(t_{k+1}) = \widehat{P}_{k+1}(t_{k+1}) + \widehat{N}_{k+1}(t_{k+1}) \widehat{W}_{k+1}(t_{k+1})^{-1} \times \widehat{N}_{k+1}(t_{k+1})^{\mathrm{T}} + w_{k+1} \widehat{C}_{1,k+1}(\beta_{k+1})^{\mathrm{T}} \widehat{C}_{1,k+1}(\beta_{k+1}).$$

The cost-to-go is

$$J_k(x) = \widehat{x}^{\mathrm{T}} \Big[\widehat{P}_k(t_k) + \widehat{N}_k(t_k) \widehat{W}_k(t_k)^{-1} \widehat{N}_k(t_k)^{\mathrm{T}} \Big] \widehat{x}.$$

Proof: By dynamic programming. See details in the appendix. □

The formulation of Proposition 3 in the compact notation (7) gives appealing formulas but in a numerical implementation it is more efficient to perform computations using a system on the form (6). For the basic trajectory planning in (1) this reduces to the following result.

Corollary 1: *Consider the optimal control problem in* (1), *where the pair* (A, B) *is controllable. The optimal Markov control in each time interval* $[t_k, t_{k+1}]$ *is*

$$u^*(t) = -R^{-1} B^{\mathrm{T}} P_k(t) X(t) - R^{-1} B^{\mathrm{T}} N_k(t) v_k - R^{-1} B^{\mathrm{T}} p_k(t)$$

with $v_k = -W_k(t_k)^{-1}(N_k(t_k)^{\mathrm{T}}X(t_k) + m_k(t_k))$, *where*

$$\begin{aligned}
&\dot{P}_k + A^{\mathrm{T}}P_k + P_kA + \Pi(P_k) = P_kBR^{-1}B^{\mathrm{T}}P_k\\
&\dot{N}_k + (A - BR^{-1}B^{\mathrm{T}}P_k)^{\mathrm{T}}N_k = 0,\\
&\dot{p}_k + (A - BR^{-1}B^{\mathrm{T}}P_k)^{\mathrm{T}}p_k = 0\\
&\dot{W}_k + N_k^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}N_k = 0\\
&\dot{m}_k = N_k^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}p_k\\
&\dot{\rho}_k = p_k^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}p_k
\end{aligned} \tag{9}$$

where $\Pi(P_k)$ *is a linear matrix function with elements* $\Pi(P)_{k,l} = \frac{1}{2}\operatorname{tr}(G_k^{\mathrm{T}}PG_l)$ *and the boundary conditions are*

$$\begin{aligned}
P_k(t_{k+1}) &= P_{k+1}(t_{k+1}) + N_{k+1}(t_{k+1})W_{k+1}(t_{k+1})^{-1}\\
&\quad \times N_{k+1}(t_{k+1})^{\mathrm{T}} + w_{k+1}C^{\mathrm{T}}C\\
p_k(t_{k+1}) &= p_{k+1}(t_{k+1}) + N_{k+1}(t_{k+1})W_{k+1}(t_{k+1})^{-1}\\
&\quad \times m_{k+1}(t_{k+1}) + w_{k+1}C^{\mathrm{T}}\alpha_{k+1}\\
\rho_k(t_{k+1}) &= \rho_{k+1}(t_{k+1}) + m_{k+1}(t_{k+1})^{\mathrm{T}}W_{k+1}(t_{k+1})^{-1}\\
&\quad \times m_{k+1}(t_{k+1}) + w_{k+1}\alpha_{k+1}^{\mathrm{T}}\alpha_{k+1}
\end{aligned}$$

and $N_k(t_{k+1}) = C^{\mathrm{T}}$, $m_k(t_{k+1}) = -\alpha_{k+1}$ *and* $W_k(t_{k+1}) = 0$.

The optimal cost-to-go is

$$\begin{aligned}
J_k(x) &= x^{\mathrm{T}}P_k(t_k)x + 2p_k(t_k)^{\mathrm{T}}x + \rho_k(t_k) + (N_k(t_k)^{\mathrm{T}}x\\
&\quad + m_k(t_k))^{\mathrm{T}}W_k(t_k)^{-1}(N_k(t_k)^{\mathrm{T}}x + m_k(t_k)).
\end{aligned}$$

3. Path planning for mobile robot

We consider the problem of steering a robot from rest at an initial condition $(-5, 1)$ to rest at the final position $(-1, 5)$ in such a way that it stays inside the corridor in the upper left part of figure 1. The dynamic model of a mobile robot with the centre of the wheel axis as a reference point will be non-linear and non-holonomic. However, by moving the reference point to an off-axis point it is possible to feedback linearize the dynamics (see e.g. Laumond 1998). We use a feedback linearization of a unicycle model derived in Lawton *et al.* (2000)

$$\ddot{y} = u + e.$$

Here we have added a noise signal e that takes into account friction, irregularities in the floor, and other error sources. If we let the components of the noise be modelled as $\mathrm{d}E_i = \dot{y}_i\,\mathrm{d}W_i$, where a W is a two-dimensional Brownian motion, then the robot dynamics can be modelled by the stochastic system

$$\begin{aligned}
\mathrm{d}X &= (AX + Bu)\,\mathrm{d}t + G(X)\,\mathrm{d}W\\
Y &= CX
\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad G(x) = \begin{bmatrix} 0 & 0\\ x_2 & 0\\ 0 & 0\\ 0 & x_4 \end{bmatrix}.$$

Let us use the design equation

$$\min E^{0,x_0}\left\{w_1|C_1X(3) - \alpha_1|^2 + w_2|C_3X(6) - \alpha_3|^2 + \int_0^6 |u|^2\mathrm{d}t\right\}$$

subject to

$$\begin{cases} \mathrm{d}X = (AX + Bu)\,\mathrm{d}t + G(X)\,\mathrm{d}W, \quad X(0) = x_0\\ E^{0,x_0}\{C_2X(3)\} = \alpha_2, \quad E^{0,x_0}\{C_3X(6)\} = \alpha_3 \end{cases} \tag{10}$$

where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} -1\\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \quad \alpha_2 = 0$$

$$C_3 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} -1\\ 0\\ 5\\ 0 \end{bmatrix}.$$

The idea behind the optimization problem is to divide the control problem into two stages. First, we steer the robot to the switching surface $C_2x = \alpha_2$ in such a way that the expected deviation from the point $C_1x = \alpha_1$ is small. Then in the next stage we steer to the final position $x = \alpha_3$ in such a way that the variance of the deviation from this point is small. The integral cost is introduced to keep the expected control energy small. With the weights $w_1 = 7$ and $w_2 = 1$ we get the result in figure 1. We see from the lower plot that the expected path of the robot stays well inside the corridor as desired. It is possible to push the trajectory further toward the middle of the corridor by adding an integral cost $E^{0,x_0}\{\int_0^6 q|Cx - y_0(t)|^2\mathrm{d}t\}$, where $q \geq 0$ and $y_0(t)$ is some nominal trajectory near the middle of the corridor. The corresponding optimization problem still belongs to the problem class considered in this paper.

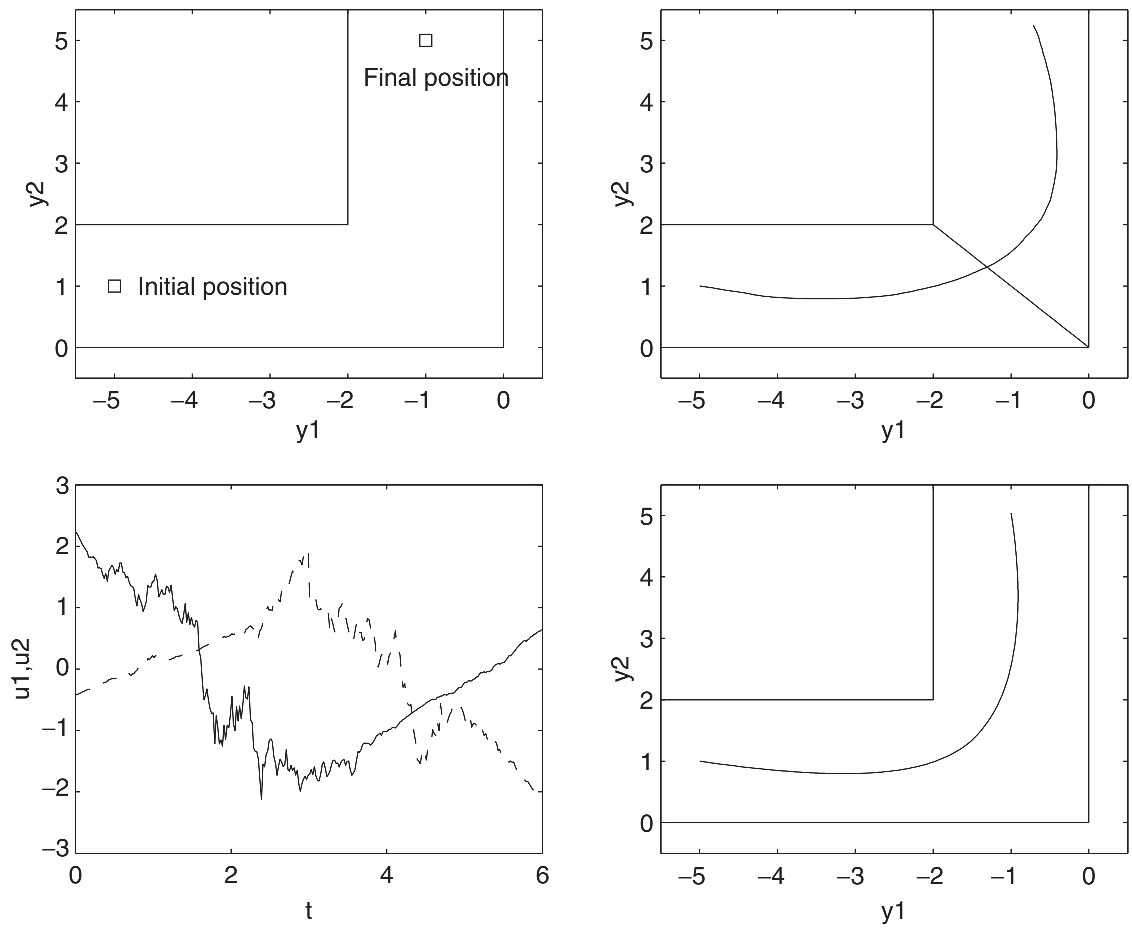

Figure 1. The upper left figure shows the initial and final positions for the robot. The upper right figure shows one realization of the optimal path of the robot and the lower right figure shows the corresponding control signals, where u_1 corresponds to the solid line and u_2 is the dashed line. The weights $w_1 = 7$ and $w_2 = 1$ were used in the optimization problem (10). Finally, the lower right figure shows an estimate of the expected path obtained by averaging over 100 stochastic simulations.

4. Interpolation with inequality constraints

In this section we consider the case with interpolation constraints on the form

$$\underline{\alpha}_k \leq E^{t_0, x_0}\{C_{2,k}X(t_k)\} \leq \overline{\alpha}_k.$$

This means that we consider

$$J_0(x_0) = \min_{u \in \mathcal{M}(t_0,t_N)} E^{t_0,x_0}\left\{\sum_{k=0}^{N-1}\left(w_{k+1}|C_{1,k+1}X(t_{k+1}) - \beta_{k+1}|^2 + \int_{t_k}^{t_{k+1}} \sigma_k(t,X,u)\,\mathrm{d}t\right)\right\}$$

$$\text{subject to} \begin{cases} \mathrm{d}X = (A_k(t)X + B_k(t)u)\,\mathrm{d}t + G_k(t,X)\,\mathrm{d}Z, \\ \quad t \in [t_k, t_{k+1}], \quad X(t_0) = x_0 \\ \underline{\alpha}_{k+1} \leq E^{t_0,x_0}\{C_{2,k+1}X(t_{k+1})\} \leq \overline{\alpha}_{k+1}, \\ \quad k = 0, \ldots, N-1. \end{cases} \tag{11}$$

The assumptions on the system matrices are the same as in the previous section. In the next result we use the compact notation of the previous section with the exception that

$$\widehat{C}_{2,k}(\alpha_k) = \begin{bmatrix} C_{2,k} & -\overline{\alpha}_k \\ -C_{2,k} & \underline{\alpha}_k \end{bmatrix}.$$

We also use the notation $\lambda_k = [\overline{\lambda}_k^{\mathrm{T}} \;\; \underline{\lambda}_k^{\mathrm{T}}]^{\mathrm{T}}$ for the Lagrange multipliers corresponding to the interpolation inequalities and let $\lambda^N = \emptyset$ and $\lambda^k = [\lambda_N^{\mathrm{T}} \;\; \ldots \;\; \lambda_{k+1}^{\mathrm{T}}]^{\mathrm{T}}$.

Proposition 4: *For $k = N-1, N-2, \ldots, 0$ let*

$$\begin{aligned} &\dot{\widehat{P}}_k + \widehat{A}_k^{\mathrm{T}}\widehat{P}_k + \widehat{P}_k\widehat{A}_k + \widehat{Q}_k + \hat{\Pi}_k(\widehat{P}_k) \\ &\quad = (\widehat{P}_k\widehat{B}_k + \widehat{S}_k)\widehat{R}_k^{-1}(\widehat{P}_k\widehat{B}_k + \widehat{S}_k)^{\mathrm{T}} \\ &\dot{\widehat{N}}_k + (\widehat{A}_k - \widehat{B}_k\widehat{R}_k^{-1}(\widehat{P}_k\widehat{B}_k + \widehat{S}_k)^{\mathrm{T}})^{\mathrm{T}}\widehat{N}_k = 0, \\ &\widehat{N}_k(t_{k+1}, \lambda^{k+1}) = \widehat{C}_{2,k+1}(\alpha_{k+1})^{\mathrm{T}} \\ &\dot{\widehat{W}}_k + \widehat{N}_k^{\mathrm{T}}\widehat{B}_k\widehat{R}_k^{-1}\widehat{B}_k^{\mathrm{T}}\widehat{N}_k = 0, \quad \widehat{W}_k(t_{k+1}) = 0 \end{aligned}$$

where $\widehat{P}_{N-1}(t_N, \lambda^N) = w_N \widehat{C}_{1,N}(\beta_N)^{\mathrm{T}} \widehat{C}_{1,N}(\beta_N)$ *and*

$$\widehat{P}_k(t_{k+1}, \lambda^{k+1}) = \bar{P}_{k+1}(t_{k+1}, \lambda^{k+1}) + w_{k+1} \widehat{C}_{1,k+1}(\beta_{k+1})^{\mathrm{T}} \widehat{C}_{1,k+1}(\beta_{k+1})$$

$$\bar{P}_k(t_k, \lambda^k) = \widehat{P}_k(t_k, \lambda^{k+1}) + e_{n+1} \lambda_{k+1}^{\mathrm{T}} \widehat{N}_k(t_k, \lambda^{k+1}) + \widehat{N}_k(t_k, \lambda^{k+1})^{\mathrm{T}} \lambda_{k+1} e_{n+1}^{\mathrm{T}} - \lambda_{k+1}^{\mathrm{T}} \widehat{W}_k(t_k) \lambda_{k+1} e_{n+1} e_{n+1}^{\mathrm{T}}$$

and where $e_{n+1} = [0 \ \cdots \ 0 \ 1] \in \mathbb{R}^{n+1}$. *Note that* $\widehat{P}_k$ *and* $\widehat{N}_k$ *depend quadratically, respectively linearly, on* λ^{k+1} *while* $\widehat{W}_k$ *is independent of* λ. *The optimal cost function is obtained as the solution to the following quadratic optimization problem*

$$\max_{\lambda^0 \geq 0} \widehat{x}_0^{\mathrm{T}} \bar{P}_0(t_0, \lambda^0) \widehat{x}_0. \tag{12}$$

If $\lambda^* = [(\lambda_N^*)^{\mathrm{T}} \ \ldots \ (\lambda_1^*)^{\mathrm{T}}]^{\mathrm{T}}$ *is the maximizing argument in* (12) *then the optimal control in each time interval* $[t_k, t_{k+1}]$ *is given as*

$$u^*(t) = \widehat{R}_k^{-1}(t)(\bar{P}_k(t, \lambda^*) \widehat{B}_k(t) + \widehat{S}_k(t))^{\mathrm{T}} \widehat{X}(t)$$

where

$$\bar{P}_k(t) = \widehat{P}_k(t, \lambda^*) + e_{n+1} (\lambda_{k+1}^*)^{\mathrm{T}} \widehat{N}_k(t, \lambda^*) + \widehat{N}_k(t, \lambda^*)^{\mathrm{T}} \lambda_{k+1}^* e_{n+1}^{\mathrm{T}} - (\lambda_{k+1}^*)^{\mathrm{T}} \widehat{W}_k(t) \lambda_{k+1}^* e_{n+1} e_{n+1}^{\mathrm{T}}.$$

Proof: See the appendix. □

5. Concluding remarks

In this paper we considered and solved the trajectory planning problem in (1) and some generalizations of it. Such problems occur in a variety of settings and there are many important application areas in which the techniques of this paper are relevant. We have already mentioned applications in trajectory planning for robots and aircrafts. Other possible applications are in population studies, and investment problems. We considered a simple example of a model for housing investments with variable interest rates in Jönsson *et al.* (2002).

There are many possible generalizations of this work. An important area is to consider problems that are of mixed types. In the simplest such examples we can assume that some data must be interpolated and some must be smoothed. These problems can be attacked using the methods of this paper. The potential applications of the techniques presented in this paper to problems in economics and finance are enormous. In this area it is interesting to consider generalizations to models where the stochastic uncertainty multiplies the control signal (Lim and Zhou 1999). Another extension is to consider the construction of splines when the dynamics are governed by two point boundary value problems (see, e.g. Adams *et al.* (1984 a, b) and Krener (1978) for techniques and examples). An interesting problem that remains open is to solve the problem when there is a constraint such as

$$E(\dot{y}(t)) \geq 0$$

for $0 \leq t \leq T$ where T is the final time. This particular problem was solved by Zhou *et al.* (2001) in the deterministic case. Both in the deterministic and the stochastic cases the problem remains open for solving the problem with constraints such as $f(t) \leq E(y(t)) \leq g(t)$ for $a < t < b$. This current paper represents an important first step in extending control theoretic splines to the stochastic setting where these problems exist. Dynamic programming is a powerful tool and fits the needs of stochastic splines very well as is demonstrated in this paper.

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Appendix

Proof of Proposition 1: Let $J^*(x)$ denote the optimal cost-to-go function in (4). We will show that it satisfies the stated dynamic programming iteration. Obviously, we have $J_N^*(x) = J_N(x) = 0$. Assume by induction that $J_{k+1}^*(x) = J_{k+1}(x)$. For compactness of notation we introduce new notation for the cost function and constraints. First, let $\bar{\alpha}_n = (\alpha_n, \ldots, \alpha_{N-1}, \alpha_N)$ and $\bar{\beta}_n = (\beta_n, \ldots, \beta_{N-1}, \beta_N)$. Then we let†

$$\mathcal{C}_n(u, \beta_{n+1}) = \int_{t_n}^{t_{n+1}} \sigma_n(t, X, u)\, \mathrm{d}t + w_{n+1} |C_{1,n+1} X(t_{n+1}) - \beta_{n+1}|^2$$

$$\mathcal{C}_n(u, \bar{\beta}_{n+1}) = \sum_{k=n}^{N-1} \left(\int_{t_k}^{t_{k+1}} \sigma_k(t, X, u)\, \mathrm{d}t + w_{k+1} |C_{1,n+1} X(t_{k+1}) - \beta_{k+1}|^2 \right)$$

$$\mathcal{U}_n(x, \bar{\alpha}_{n+1}) = \left\{ u \in \mathcal{M}(t_n, t_N) \colon \left\{ \begin{array}{l} \mathrm{d}X = (A_k X + B_k u)\, \mathrm{d}t \\ \quad + G_k(X)\, \mathrm{d}Z, \quad t \in [t_k, t_{k+1}], \\ \quad X(t_n) = x \\ E^{t_n, x}\{C_{2,k+1} X(t_{k+1})\} = \alpha_{k+1}, \\ \quad k = n, \ldots N-1 \end{array} \right\} \right\}$$

†In the definition of $\mathcal{C}_n(u, \beta_{n+1})$ and $\mathcal{C}_n(u, \bar{\beta}_{n+1})$ we implicitly assume that $X(t)$ satisfies the stochastic differential equations in the definition of $\mathcal{U}_n(x, \alpha_{n+1})$ and $\mathcal{U}_n(x, \bar{\alpha}_{n+1})$, respectively.

and finally $\mathcal{U}_n(x, \alpha_{n+1})$ is defined similarly but only on the time interval $[t_n, t_{n+1}]$. We have

$$
\begin{aligned}
J_n^*(x) &= \min_{u\in\mathcal{U}_n(x,\bar{\alpha}_{n+1})} E^{t_n,x}\left\{\sum_{k=n}^{N-1}\left(\int_{t_k}^{t_{k+1}}\sigma_k(t,X,u)\,\mathrm{d}t + w_{k+1}|C_{1,k+1}X(t_{k+1})-\beta_{k+1}|^2\right)\right\}\\
&= \min_{u\in\mathcal{U}_n(x,\bar{\alpha}_{n+1})} E^{t_n,x}\mathcal{C}_n(u,\bar{\beta}_{n+1})\\
&= \min_{u_1\in\mathcal{U}_n(x,\alpha_{n+1})} E^{t_n,x}\Big\{\mathcal{C}_n(u_1,\beta_{n+1}) + \min_{u_2\in\mathcal{U}_{n+1}(X(t_{n+1}),\bar{\alpha}_{n+2})} E^{t_n,x}\{\mathcal{C}_{n+1}(u_2,\bar{\beta}_{n+2})\,|\,X(t_{n+1})\}\Big\}\\
&= \min_{u_1\in\mathcal{U}_n(x,\alpha_{n+1})} E^{t_n,x}\Big\{\mathcal{C}_n(u_1,\beta_{n+1}) + \min_{u_2\in\mathcal{U}_{n+1}(X(t_{n+1}),\bar{\alpha}_{n+2})} E^{t_{n+1},X(t_{n+1})}\{\mathcal{C}_{n+1}(u_2,\bar{\beta}_{n+2})\}\Big\}\\
&= \min_{u\in\mathcal{U}_n(x,\alpha_{n+1})} E^{t_n,x}\{\mathcal{C}_n(u,\beta_{n+1}) + J_{n+1}^*(X(t_{n+1}))\}\\
&= \min_{u\in\mathcal{U}_n(x,\alpha_{n+1})} E^{t_n,x}\Big\{\mathcal{C}_n(u,\beta_{n+1}) + J_{n+1}(X(t_{n+1}))\Big\}\\
&= J_n(x)
\end{aligned}
$$

where the Markov property was used in the third equality and the induction hypothesis in the fourth.

□

Proof of Proposition 2: We use the compact notation introduced in (7) and (8) after the proposition. If we apply the Lagrange multiplier rule to (8) then we obtain

$$
\begin{aligned}
&V(x_0,\alpha,t_0,t_f)\\
&= \max_{\lambda}\min_{u\in\mathcal{M}(t_0,t_f)} E^{t_0,x_0}\left\{\int_0^{t_f}[\widehat{X}^{\mathrm{T}}\widehat{Q}\widehat{X} + 2\widehat{X}^{\mathrm{T}}\widehat{S}u + u^{\mathrm{T}}\widehat{R}u]\,\mathrm{d}t + \widehat{X}(t_f)^{\mathrm{T}}\widehat{Q}_0\widehat{X}(t_f) + 2\lambda^{\mathrm{T}}\widehat{C}(\alpha)\widehat{X}(t_f)\right\}\\
&\quad \text{subject to}\quad \mathrm{d}\widehat{X} = (\widehat{A}(t)\widehat{X} + \widehat{B}(t)u)\,\mathrm{d}t + \widehat{G}(t,X)\,\mathrm{d}Z,\\
&\qquad \widehat{X}(t_0) = \widehat{x}_0.
\end{aligned}
$$

The solution to the inner optimization can be obtained from the Hamilton–Jacobi–Bellman equation (HJBE)

$$
\begin{aligned}
-\frac{\partial V}{\partial t} &= \min_{u\in\mathbb{R}^m}\left\{\widehat{x}^{\mathrm{T}}\widehat{Q}\widehat{x} + 2\widehat{x}^{\mathrm{T}}\widehat{S}u + u^{\mathrm{T}}\widehat{R}u + \frac{\partial V}{\partial \widehat{x}}^{\mathrm{T}}(\widehat{A}\widehat{x} + \widehat{B}u) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j}\right\}\\
V(x,t_f) &= \widehat{x}^{\mathrm{T}}\widehat{Q}_0\widehat{x} + 2\lambda^{\mathrm{T}}\widehat{C}(\alpha)\widehat{x}
\end{aligned}
$$

where

$$
\begin{aligned}
a_{ij} = (\sigma\sigma^{\mathrm{T}})_{ij} &= \left(\left(\sum_{k=1}^{n} x_k G_k\right)\left(\sum_{l=1}^{n} x_l G_l^{\mathrm{T}}\right)\right)_{ij}\\
&= \sum_{k,l=1}^{n} x_k x_l G_{ki} G_{lj}^{\mathrm{T}}
\end{aligned}
$$

where G_{ki} is the ith row of G_k. With the value function $V(\widehat{x},t) = \widehat{x}^{\mathrm{T}}\bar{P}(t)\widehat{x}$ we get

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j}\\
&\quad = \frac{1}{2}\sum_{k=1}^{n}\sum_{l=1}^{n} x_k x_l \operatorname{tr}(G_k^{\mathrm{T}} P G_l) = \widehat{x}^{\mathrm{T}}\hat{\Pi}(\bar{P})\widehat{x}.
\end{aligned}
$$

If we plug $V(\widehat{x},t) = \widehat{x}^{\mathrm{T}}\bar{P}(t)\widehat{x}$ into the HJBE we get the optimal control $u = -\widehat{R}^{-1}(\bar{P}\widehat{B} + \widehat{S})^{\mathrm{T}}\widehat{x}$, and that the following Riccati equation must hold

$$
\dot{\bar{P}} + \widehat{A}^{\mathrm{T}}\bar{P} + \bar{P}\widehat{A} + \widehat{Q} + \hat{\Pi}(\bar{P}) = (\bar{P}\widehat{B} + \widehat{S})\widehat{R}^{-1}(\bar{P}\widehat{B} + \widehat{S})^{\mathrm{T}}
$$

with boundary condition $\bar{P}(t_f) = \widehat{Q}_0 + e_{n+1}\lambda^{\mathrm{T}}\widehat{C} + \widehat{C}^{\mathrm{T}}\lambda e_{n+1}^{\mathrm{T}}$, where $e_{n+1} = [0 \quad \cdots \quad 0 \;\; 1]^{\mathrm{T}} \in \mathbb{R}^{n+1}$. The optimal cost becomes

$$
V(x_0,\alpha,t_0,t_f) = \max_{\lambda}\widehat{x}_0^{\mathrm{T}}\bar{P}(\lambda,t_0)\widehat{x}_0. \tag{13}
$$

To perform the optimization with respect to λ we need to obtain an explicit expression of $\bar{P}$ as a function of λ. It turns out that we can use

$$
\bar{P} = \widehat{P} + \widehat{N}\lambda\, e_{n+1}^{\mathrm{T}} + e_{n+1}\lambda^{\mathrm{T}}\widehat{N}^{\mathrm{T}} - \lambda^{\mathrm{T}}\widehat{W}\lambda e_{n+1}e_{n+1}^{\mathrm{T}}
$$

where $\widehat{P}, \widehat{N}$ and $\widehat{W}$ are given in (7). This follows since

$$
\begin{aligned}
\bar{P}(t_f) &= \widehat{P}(t_f) + \widehat{N}(t_f)\lambda e_{n+1}^{\mathrm{T}} + e_{n+1}\lambda^{\mathrm{T}}\widehat{N}(t_f)^{\mathrm{T}}\\
&\quad - \lambda^{\mathrm{T}}\widehat{W}(t_f)\lambda e_{n+1}e_{n+1}^{\mathrm{T}}\\
&= \widehat{Q}_0 + e_{n+1}\lambda^{\mathrm{T}}\widehat{C} + \widehat{C}^{\mathrm{T}}\lambda e_{n+1}^{\mathrm{T}}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\bar{P}} &= -\widehat{A}^{\mathrm{T}}\widehat{P} - \widehat{P}\widehat{A} - \widehat{Q} - \hat{\Pi}(\bar{P}) + (\widehat{P}\widehat{B} + \widehat{S})\widehat{R}^{-1}(\widehat{P}\widehat{B} + \widehat{S})^{\mathrm{T}}\\
&\quad - (\widehat{A} - \widehat{B}\widehat{R}^{-1}(\widehat{B}^{\mathrm{T}}\widehat{P} + \widehat{S}^{\mathrm{T}}))^{\mathrm{T}}\widehat{N}\lambda e_{n+1}^{\mathrm{T}}\\
&\quad - e_{n+1}\lambda^{\mathrm{T}}\widehat{N}^{\mathrm{T}}(\widehat{A} - \widehat{B}\widehat{R}^{-1}(\widehat{B}^{\mathrm{T}}\widehat{P} + \widehat{S}^{\mathrm{T}}))\\
&\quad - \lambda^{\mathrm{T}}\widehat{N}^{\mathrm{T}}\widehat{B}\widehat{R}^{-1}\widehat{B}^{\mathrm{T}}\widehat{N}\lambda e_{n+1}e_{n+1}^{\mathrm{T}}\\
&= -\widehat{A}^{\mathrm{T}}\bar{P} - \bar{P}\widehat{A} - \widehat{Q} - \hat{\Pi}(\bar{P}) + (\bar{P}\widehat{B} + \widehat{S})\widehat{R}^{-1}(\bar{P}\widehat{B} + \widehat{S})^{\mathrm{T}}
\end{aligned}
$$

which follows since

$$
\begin{aligned}
\widehat{A}^{\mathrm{T}}(e_{n+1}\lambda^{\mathrm{T}}\widehat{N}^{\mathrm{T}} + \lambda^{\mathrm{T}}\widehat{W}\lambda e_{n+1}e_{n+1}^{\mathrm{T}}) &= 0\\
\widehat{B}^{\mathrm{T}}(e_{n+1}\lambda^{\mathrm{T}}\widehat{N}^{\mathrm{T}} + \lambda^{\mathrm{T}}\widehat{W}\lambda e_{n+1}e_{n+1}^{\mathrm{T}}) &= 0.
\end{aligned}
$$

The optimization in (13) thus becomes

$$\max_{\lambda} \hat{x}_0^{\mathrm{T}}(\hat{P}(t_0) + \hat{N}(t_0)\lambda e_{n+1}^{\mathrm{T}} + e_{n+1}\lambda^{\mathrm{T}}\hat{N}(t_0)^{\mathrm{T}} - \lambda^{\mathrm{T}}\hat{W}(t_0)\lambda e_{n+1}e_{n+1}^{\mathrm{T}})\hat{x}_0 = \hat{x}_0^{\mathrm{T}}(\hat{P}(t_0) + \hat{N}(t_0)\hat{W}(t_0)^{-1}\hat{N}(t_0)^{\mathrm{T}})\hat{x}_0$$

and the optimal Lagrange multiplier is $\lambda = \hat{W}(t_0)^{-1}\hat{N}(t_0)^{\mathrm{T}}\hat{x}_0$. Finally, the optimal control becomes

$$u = -\hat{R}^{-1}(\hat{B}^{\mathrm{T}}(\hat{P} + \hat{N}\hat{W}(t_0)^{-1}\hat{N}(t_0)^{\mathrm{T}}\hat{x}_0 e_{n+1}^{\mathrm{T}}) + \hat{S})\hat{x} = -\hat{R}^{-1}(\hat{P}\hat{B} + \hat{S})^{\mathrm{T}}\hat{x} - \hat{R}^{-1}\hat{B}^{\mathrm{T}}\hat{N}\hat{W}(t_0)^{-1}\hat{N}(t_0)^{\mathrm{T}}\hat{x}_0.$$

It is straightforward to show that $E^{t_0,x_0}\{\hat{C}(\alpha)\hat{X}(t_f)\} = 0$ when using this control, i.e. the constraint is satisfied. This proves that we have obtained an optimal solution (Øksendal 1998). Considering the optimal control problem from the 'initial point' $(t, x(t))$ gives the following feedback formulation of the optimal control $u = -\hat{R}^{-1}((\hat{P} + \hat{N}\hat{W}^{-1}\hat{N}^{\mathrm{T}})\hat{B} + \hat{S})^{\mathrm{T}}\hat{X}$. We note that under the conditions of the proposition there exist solutions $\bar{P}$ and $\hat{P}$ to the Riccati equations that are involved in the proof. Indeed, the special structure of the system matrices implies that only the upper left blocks of $\bar{P}$ and $\hat{P}$ satisfy nonlinear equations, which are identical to the first equation in (6). The other blocks corresponds to p and ρ in (6). Hence, the existence of $\bar{P}$ and $\hat{P}$ follows from Remark 1. □

Proof of Proposition 3: The dynamic programming recursion is

$$J_k(x) = \min_u E^{t_k,x}\left\{\int_{t_k}^{t_{k+1}} [\hat{X}^{\mathrm{T}}\hat{Q}_k\hat{X} + 2\hat{X}^{\mathrm{T}}\hat{S}_k u + u^{\mathrm{T}}\hat{R}_k u]\,\mathrm{d}t + w_{k+1}|\hat{C}_{1,k+1}(\beta_{k+1})\hat{X}(t_{k+1})|^2 + J_{k+1}(X(t_{k+1}))\right\}$$

$$\text{subject to} \begin{cases} \mathrm{d}\hat{X} = (\hat{A}_k(t)\hat{X} + \hat{B}_k(t)u)\,\mathrm{d}t + \hat{G}_k(t,X)\,\mathrm{d}Z, \quad \hat{X}(t_k) = \hat{x} \\ E^{t_k,x}\{\hat{C}_{2,k+1}(\alpha_{k+1})\hat{X}(t_{k+1})\} = 0 \end{cases}$$

$$J_N(\hat{x}) = 0.$$

It follows from the discussion preceding the proposition that

$$J_{N-1}(x) = \hat{x}^{\mathrm{T}}(\hat{P}_{N-1}(t_{N-1}) + \hat{N}_{N-1}(t_{N-1})\hat{W}_{N-1}(t_{N-1})^{-1}\hat{N}_{N-1}(t_{N-1})^{\mathrm{T}})\hat{x}$$

where all variables are defined as in the proposition and $\hat{P}_{N-1}(t_N) = w_N\hat{C}_{1,N}(\beta_N)^{\mathrm{T}}\hat{C}_{1,N}(\beta_N)$. In the next recursion we get an analogous result except that now the boundary condition will be

$$\hat{P}_{N-2}(t_{N-1}) = w_{N-1}\hat{C}_{1,N-1}(\beta_{N-1})^{\mathrm{T}}\hat{C}_{1,N-1}(\beta_{N-1}) + \hat{P}_{N-1}(t_{N-1}) + \hat{N}_{N-1}(t_{N-1}) \times \hat{W}_{N-1}(t_{N-1})^{-1}\hat{N}_{N-1}(t_{N-1})^{\mathrm{T}}.$$

It is now obvious that the recursion continues as in the proposition statement. □

Proof of Proposition 4: Lagrange relaxation of the inequality constraints gives in our compact notation

$$J_0(x_0) = \max_{\lambda^0 \geq 0} \min_{u \in \mathcal{M}(t_0,t_N)} E^{t_0,x_0}\left\{\sum_{k=0}^{N-1}\left(w_{k+1}|C_{1,k+1}X(t_{k+1}) - \beta_{k+1}|^2 + \lambda_{k+1}^{\mathrm{T}}\hat{C}_{2,k+1}(\alpha_{k+1})\hat{X}(t_{k+1}) + \int_{t_k}^{t_{k+1}}\left[\hat{X}^{\mathrm{T}}\hat{Q}_k\hat{X} + 2\hat{X}^{\mathrm{T}}\hat{S}_k u + u^{\mathrm{T}}\hat{R}_k u\right]\mathrm{d}t\right)\right\}$$

$$\text{subject to} \quad \mathrm{d}\hat{X} = (\hat{A}_k(t)\hat{X} + \hat{B}_k(t)u)\,\mathrm{d}t + G_k(t,X)\,\mathrm{d}Z, \quad t \in [t_k, t_{k+1}], \quad \hat{X}(t_0) = \hat{x}_0.$$

We use a dynamic programming iteration with $J_N(x) = 0$. The derivation in the proof of Proposition 2 gives at $t = t_{N-1}$

$$J_{N-1}(x, \lambda^{N-1}) = \max_{\lambda_N \geq 0} \hat{x}^{\mathrm{T}}(\hat{P}_{N-1}(t_{N-1}, \lambda^N) + \hat{N}_{N-1}(t_{N-1}, \lambda^N)\lambda_N e_{n+1}^{\mathrm{T}} + e_{n+1}\lambda_N^{\mathrm{T}}\hat{N}_{N-1}(t_{N-1}, \lambda^N)^{\mathrm{T}} - \lambda_N^{\mathrm{T}}\hat{W}_{N-1}(t_{N-1})\lambda_N e_{n+1}e_{n+1}^{\mathrm{T}})\hat{x} =: \max_{\lambda^{N-1} \geq 0} \hat{x}^{\mathrm{T}}\bar{P}_{N-1}(t_{N-1}, \lambda^{N-1})\hat{x}.$$

In the next iteration we get

$$\begin{aligned} J_{N-2}(x,\lambda^{N-2}) &= \max_{\lambda_{N-1} \geq 0} \min_{u \in \mathcal{M}(t_{N-2},t_{N-1})} E^{t_{N-2},x}\Big\{w_{N-1}|\hat{C}_{1,N-1}(\beta_{N-1})\hat{X}(t_{N-1})|^2 + \lambda_{N-1}^{\mathrm{T}}\hat{C}_{2,N-1}(\alpha_{N-1})\hat{X}(t_{N-1}) \\ &\quad + \int_{t_{N-2}}^{t_{N-1}}\left[\hat{X}^{\mathrm{T}}\hat{Q}_k\hat{X} + 2\hat{X}^{\mathrm{T}}\hat{S}_k u + u^{\mathrm{T}}\hat{R}_k u\right]\mathrm{d}t + \max_{\lambda^{N-1} \geq 0} \hat{X}(t_{N-1})^{\mathrm{T}}\bar{P}_{N-1}(t_{N-1}, \lambda^{N-1})\hat{X}(t_{N-1})\Big\} \\ &= \max_{\substack{\lambda^{N-1} \geq 0 \\ \lambda_{N-1} \geq 0}} \min_{u \in \mathcal{M}(t_{N-2},t_{N-1})} E^{t_{N-2},x}\Big\{w_{N-1}|\hat{C}_{1,N-1}(\beta_{N-1})\hat{X}(t_{N-1})|^2 + \hat{X}(t_{N-1})^{\mathrm{T}}\bar{P}_{N-1}(t_{N-1},\lambda^{N-1})\hat{X}(t_{N-1}) \\ &\quad + \lambda_{N-1}^{\mathrm{T}}\hat{C}_{2,N-1}(\alpha_{N-1})\hat{X}(t_{N-1}) + \int_{t_{N-2}}^{t_{N-1}}\left[\hat{X}^{\mathrm{T}}\hat{Q}_k\hat{X} + 2\hat{X}^{\mathrm{T}}\hat{S}_k u + u^{\mathrm{T}}\hat{R}_k u\right]\mathrm{d}t\Big\} \\ &=: \max_{\lambda^{N-2} \geq 0} \hat{x}^{\mathrm{T}}\bar{P}_{N-2}(t_{N-2},\lambda^{N-2})\hat{x} \end{aligned}$$

where

$$\begin{aligned}
\bar{P}_{N-2}(t_{N-2}, \lambda^{N-2}) &\\
= \widehat{P}_{N-2}(t_{N-2}, \lambda^{N-1}) &+ \widehat{N}_{N-2}(t_{N-2}, \lambda^{N-1})\lambda_{N-1}e_{n+1}^{\mathrm{T}} \\
&+ e_{n+1}\lambda_{N-1}^{\mathrm{T}}\widehat{N}_{N-2}(t_{N-2}, \lambda^{N-1})^{\mathrm{T}} \\
&- \lambda_{N-1}^{\mathrm{T}}\widehat{W}_{N-2}(t_{N-2})\lambda_{N-1}e_{n+1}e_{n+1}^{\mathrm{T}}.
\end{aligned}$$

The second equality, where $\min_{u\in\mathcal{M}(t_{N-2}, t_{N-1})}$ and $\max_{\lambda^{N-1}}$ are permuted follows from the Karush-Kuhn–Tucker theorem (Balakrishnan 1976). Indeed, the optimization problem involves a convex cost and convex constraints so the Lagrange function satisfies a saddle-point condition, which implies that the min and the max commutes. In the third equality we used the same arguments as in the proof of Proposition 2. If we continue the recursion we obtain (12).

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