Homework for
Short Course on Sampled-Data Control Theory

Hisaya Fujioka

Kyoto University, visiting KTH
fujioka@{i.kyoto-u.ac.jp,math.kth.se}

1. Derive a finite-dimensional state-space representation of the discrete-time system below, mapping u_d to y_d, where G_c is a continuous-time system with state-space representation

$$\begin{bmatrix} \dot{x}_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ v_c(t) \end{bmatrix}.$$

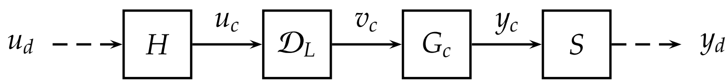
   S and H are an ideal sampler with sampling period $h > 0$ and a zero order hold respectively:

$$S : y_c \mapsto y_d;\ y_d[k] = y_c(kh), \tag{1}$$

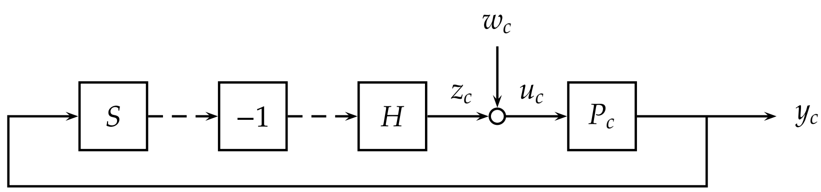
$$H : u_d \mapsto u_c;\ u_c(kh + \theta) = u_d[k],\ \theta \in [0, h). \tag{2}$$

   Finally $\mathcal{D}_L$ is a time delay of length $L \in (0, h)$:

$$\mathcal{D}_L : u_c \mapsto v_c;\ v_c(t) = u_c(t - L).$$



2. Complete the proof for the LMI condition for the feasibility of the discrete-time $\mathbf{H}_\infty$ control synthesis problem, which we have studied in Lecture #3.
3. Consider the sampled-data system shown below:



   P_c is a plant, and is an integrator with a state-space realization

$$\begin{bmatrix} \dot{y}_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_c(t) \\ u_c(t) \end{bmatrix}.$$

S and H are an ideal sampler and a zero order hold as defined in (1) and (2) respectively.

(a) Prove that the sampled-data system is internally stable if the sampling period h satisfies $0 < h < 2$.

(b) Define an operator $\grave{C}\colon \mathbb{R}^n \to \mathbf{L}_2([0, h], \mathbb{R}^p)$ by

$$\mathbb{R}^n \ni x \mapsto (\grave{C}x)(t) := Ce^{At}x.$$

where $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$. Show that

$$\grave{C}^*\grave{C} = \int_0^h e^{A^\top t}C^\top Ce^{At}\,dt. \tag{3}$$

(c) Let $w_c = 0$ and $y_c(0) = 1$. Plot $(h, \left\|y_c\right\|_2)$ for $h \in (0, 2)$.

(d) Plot $(h, \|T\|_{\mathbf{L}_2\to\mathbf{L}_2})$ for $h \in (0, 2)$, where $\|T\|_{\mathbf{L}_2\to\mathbf{L}_2}$ denotes the $\mathbf{L}_2$-induced norm from w_c to z_c.