Review of Discrete-Time Control Theory

Lecture #2

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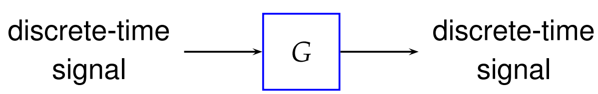
Outline

- Time Domain Model
  - State-Space Model
  - Stability
  - Stabilizability and Detectability
- Frequency Domain Model
  - $\mathcal{Z}$-Transformation
  - Transfer Function
  - Frequency Response
- LQ Optimal Control
  - Problem Formulation
  - Discrete-Time ARE
  - LQ Optimal Control

Discrete-Time System

Discrete-time system



Discrete-time signal: sequence

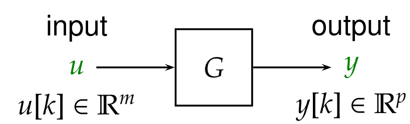
- Underlying "time": discrete
  - $\mathbb{Z}_+ := \{0, 1, 2, 3, \ldots\}$
  - $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- Taking values
  - Real vector
  - Complex vector
- Notation: $x[k]$

State-Space Model

State-space model for

- causal
- finite-dimensional
- linear, and
- time-invariant

discrete-time systems:



$$\begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

$$A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}$$

Internal Stability

Solution to $x[k+1] = Ax[k] + Bu[k]$:

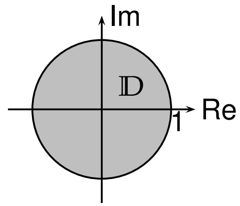
$$x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} B u[i]$$

Theorem Fix $u[k] \equiv 0$

$x[k] \to 0 \quad (k \to \infty), \quad {}^{\forall} x[0]$

$\Leftrightarrow \quad \mathrm{eig}(A) \subset \mathbb{D}$

$\Leftrightarrow \quad \rho(A) < 1$



Note: $\exists$ various definitions of stability

Stabilizability

Definition (A, B) is stabilizable

$\overset{\text{def}}{\Longleftrightarrow} \quad {}^{\exists} F \in \mathbb{R}^{m \times n}$ such that $\mathrm{eig}(A + BF) \subset \mathbb{D}$

Theorem SAE

- (A, B) is stabilizable
- $\mathrm{rank} \begin{bmatrix} zI - A & B \end{bmatrix} = n, \quad {}^{\forall} z \in \mathbb{C} \setminus \mathbb{D}$
- $\mathrm{rank} \begin{bmatrix} zI - A & B \end{bmatrix} = n, \quad {}^{\forall} z \in \{ z : z \in \mathrm{eig}(A), z \notin \mathbb{D} \}$

Reachability and Controllability

Definition (A, B) is reachable

$\overset{\text{def}}{\Longleftrightarrow}$ Fix $x[0] = 0$. ${}^{\exists} \ell$, $\{u[k]\}_{k=0}^{\ell}$ such that $x[\ell] = x_1$, ${}^{\forall} x_1$

Definition (A, B) is controllable

$\overset{\text{def}}{\Longleftrightarrow} \quad {}^{\exists} \ell$, $\{u[k]\}_{k=0}^{\ell}$ such that $x[\ell] = 0$, ${}^{\forall} x[0]$

- Continuous-time system: reachability $\Leftrightarrow$ controllability
- Discrete-time system: reachability $\Rightarrow$ controllability
- Discrete-time system with nonsingular A: reachability $\Leftrightarrow$ controllability

Detectability

Definition (A, B) is detectable

$\overset{\text{def}}{\Longleftrightarrow} \quad {}^{\exists} L \in \mathbb{R}^{n \times p}$ such that $\mathrm{eig}(A + LC) \subset \mathbb{D}$

Theorem SAE

- (A, C) is detectable
- $\mathrm{rank} \begin{bmatrix} zI - A \\ C \end{bmatrix} = n, \quad {}^{\forall} z \in \mathbb{C} \setminus \mathbb{D}$
- $\mathrm{rank} \begin{bmatrix} zI - A \\ C \end{bmatrix} = n, \quad {}^{\forall} z \in \{ z : z \in \mathrm{eig}(A), z \notin \mathbb{D} \}$

Outline

$\mathcal{Z}$-Transformation

Definition Given discrete-time signal f

$$\hat{f}[z] = \mathcal{Z}\{f\}[z] := \sum_{k=0}^{\infty} f[k]z^{-k}$$

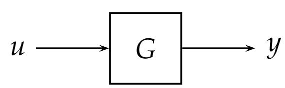
- Example: discrete-time impulse $\delta[k] = \begin{cases} 1, & k = 0 \\ 0, & \text{o.e.} \end{cases}$

$$\mathcal{Z}\{\delta\}[z] = 1$$

- Property: Given x, y; $y[k] = x[k-1]$ (backward shift)

$$\hat{y}[z] = z^{-1}\hat{x}[z]$$

Transfer Function



$$\hat{y}[z] = \hat{G}[z]\hat{u}[z]$$

Property: Given SS model

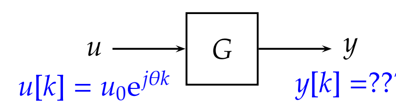
$$\begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

Transfer function is

$$\hat{G}[z] = C(zI - A)^{-1}B + D$$

Notation: Symbol with hat denotes tranfer function

Frequency Response



- Response to sinusoidal input u: $u[k] = u_0 e^{j\theta k}$

$$\begin{aligned} y[k] &= CA^k x[0] + C\sum_{\ell=0}^{k-1} A^{k-1-\ell}Bu_0 e^{j\theta\ell} + Du_0 e^{j\theta k} \\ &= \hat{G}[e^{j\theta}]u_0 e^{j\theta k} + CA^k\left(x[0] - (e^{j\theta}I - A)^{-1}Bu_0\right) \end{aligned}$$

where $\hat{G}[z] = C(zI - A)^{-1}B + D$

- Claim: $y[k] \approx \hat{G}[e^{j\theta}]u_0 e^{j\theta k}$ if $\text{eig}(A) \subset \mathbb{D}$ and $k \gg 1$
- Frequency response: $\hat{G}[e^{j\theta}]$, $\theta \in [0, 2\pi)$

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LQ Optimal Control Problem

Problem

- Given:

$$x[k+1] = Ax[k] + Bu[k], \quad x[0] = x_0$$

- Find: u which minimizes

$$J := \sum_{k=0}^{\infty} z^{\mathsf{T}}[k]z[k], \quad z[k] := Cx[k] + Du[k]$$

Discrete-Time ARE

Discrete-Time Algebraic Riccati Equation (DARE)

Given $(A, B, M) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{(n+m)\times(n+m)}$

$$A^{\mathsf{T}}PA - P + Q - (A^{\mathsf{T}}PB + S)(B^{\mathsf{T}}PB + R)^{-1}(B^{\mathsf{T}}PA + S^{\mathsf{T}}) = 0$$

$$\begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} := M$$

Stabilizing Solution

$P \in \mathbb{R}^{n\times n}$ is a stabilizing solution to DARE if

- P is a solution to DARE
- $\mathrm{eig}(A + BF) \subset \mathbb{D}, \quad F := -(B^{\mathsf{T}}PB + R)^{-1}(B^{\mathsf{T}}PA + S^{\mathsf{T}})$

Discrete-Time Lyapunov Equation

Discrete-Time Lyapunov Equation: A (square), $Q = Q^{\mathsf{T}}$

$$A^{\mathsf{T}}XA - X + Q = 0$$

Lemma Suppose $\mathrm{eig}(A) \subset \mathbb{D}$

$$X = \sum_{i=0}^{\infty} \left(A^{\mathsf{T}}\right)^i QA^i$$

Corollary $Q \geq 0 \quad \Rightarrow \quad X \geq 0$

Properties of Stabilizing Solution (1/2)

Theorem Suppose $^\exists P$: stabilizing solution to DARE

- P is unique
- $P = P^{\mathsf{T}}$
- $M \geq 0 \;\Rightarrow\; P \geq 0$

Sketch of Proof:
(Symmetry) Substituting F into DARE

$$(A + BF)^{\mathsf{T}} P (A + BF) - P + Q + SF + F^{\mathsf{T}} S^{\mathsf{T}} + F^{\mathsf{T}} R F = 0$$

This implies $\quad (A + BF)^{\mathsf{T}} (P - P^{\mathsf{T}})(A + BF) - (P - P^{\mathsf{T}}) = 0$

Noting $\mathrm{eig}(A + BF) \subset \mathbb{D}$, Lemma implies $P - P^{\mathsf{T}} = 0$

Properties of Stabilizing Solution (2/2)

(Uniqueness) Let $P_1 = P_1^{\mathsf{T}}$, $P_2 = P_2^{\mathsf{T}}$: stabilizing solutions
Some manipulation implies

$$P_1 - P_2 = (A + BF_1)^{\mathsf{T}} (P_1 - P_2)(A + BF_2)$$

$$P_1 - P_2 = \left((A + BF_1)^{\mathsf{T}}\right)^n (P_1 - P_2)(A + BF_2)^n$$

$n \to \infty$ implies $P_1 - P_2 = 0$

(SPD-ness) Manipulation implies

$$(A + BF)^{\mathsf{T}} P (A + BF) - P + \begin{bmatrix} I & F^{\mathsf{T}} \end{bmatrix} M \begin{bmatrix} I \\ F \end{bmatrix} = 0$$

Invoking Lemma, $M \geq 0$ implies $P \geq 0$

Existence of Stabilizing Solution

Theorem Given (A, B, C, D) such that

- (A, B): stabilizable
- $\begin{bmatrix} e^{j\theta} I - A & B \\ C & D \end{bmatrix}$: column full rank, $^\forall \theta \in [0, 2\pi)$

$^\exists$ stabilizing solution to

$$A^{\mathsf{T}} P A - P + C^{\mathsf{T}} C - (A^{\mathsf{T}} P B + C^{\mathsf{T}} D)(B^{\mathsf{T}} P B + D^{\mathsf{T}} D)^{-1} (B^{\mathsf{T}} P A + D^{\mathsf{T}} C) = 0$$

Comment: Parallel proof to CARE is not available at present
One good way is to cast to CARE by bilinear transformation

(cf. Lecture #3)

LQ Optimal Control Problem

Problem

- Given:

$$x[k+1] = Ax[k] + Bu[k], \quad x[0] = x_0$$

- Find: u which minimizes

$$J := \sum_{k=0}^{\infty} z^{\mathsf{T}}[k] z[k], \quad z[k] := Cx[k] + Du[k]$$

LQ Optimal Control

Assumption Given (A, B, C, D) such that

- (A, B): stabilizable
- $\begin{bmatrix} e^{j\theta} I - A & B \\ C & D \end{bmatrix}$: column full rank, $^\forall \theta \in [0, 2\pi)$

Theorem LQ optimal control is

$$u[k] = Fx[k], \quad F := -(B^\top PB + D^\top D)^{-1}(B^\top PA + D^\top C)$$

P is a stabilizing solution to DARE

$$A^\top PA - P + C^\top C - (A^\top PB + C^\top D)(B^\top PB + D^\top D)^{-1}(B^\top PA + D^\top C) = 0$$

Proof of Theorem (1/2)

Substitute DARE into J

$$\begin{aligned}
& (Cx[k] + Du[k])^\top (Cx[k] + Du[k]) \\
= \ & x^\top[k]C^\top Cx[k] + x^\top[k]C^\top Du[k] + u^\top[k]D^\top Cx[k] + u^\top[k]D^\top Du[k] \\
= \ & x^\top[k]\Big(P - A^\top PA + (A^\top PB + C^\top D)(B^\top PB + D^\top D)^{-1}(B^\top PA + D^\top C)\Big)x[k] \\
& + x^\top[k]C^\top Du[k] + u^\top[k]D^\top Cx[k] + u^\top[k]D^\top Du[k] \\
= \ & v^\top[k](B^\top PB + D^\top D)v[k] - u^\top[k]B^\top PBu[k] \\
& - u^\top[k](B^\top PA + D^\top C)x[k] - x^\top[k](A^\top PB + C^\top D)u[k] \\
& + x^\top[k](P - A^\top PA)x[k] + x^\top[k]C^\top Du[k] + u^\top[k]D^\top Cx[k]
\end{aligned}$$

where $v[k] := u[k] + (B^\top PB + D^\top D)^{-1}(B^\top PA + D^\top C)x[k]$

Proof of Theorem (2/2)

$$\begin{aligned}
& (Cx[k] + Du[k])^\top (Cx[k] + Du[k]) \\
= \ & v^\top[k](B^\top PB + D^\top D)v[k] \\
& + x^\top[k]Px[k] - (Ax[k] + Bu[k])^\top P(Ax[k] + Bu[k]) \\
= \ & v^\top[k](B^\top PB + D^\top D)v[k] \\
& + x^\top[k]Px[k] - x^\top[k+1]Px[k+1]
\end{aligned}$$

$$\begin{aligned}
J &= \sum_{k=0}^{\infty} (Cx[k] + Du[k])^\top (Cx[k] + Du[k]) \\
&= \sum_{k=0}^{\infty} v^\top[k](B^\top PB + D^\top D)v[k] + x_0^\top Px_0 - \lim_{k\to\infty} x^\top[k]Px[k]
\end{aligned}$$