Discrete-Time $\mathbf{H}_\infty$ Control Synthesis

Lecture #3

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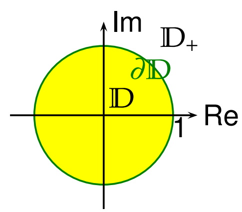
Outline

$\mathbf{H}_\infty$ Norm (1/2)

Notation

- $\mathbb{D} := \{\, z \mid z \in \mathbb{C},\ |z| < 1\}$
- $\partial\mathbb{D} := \{\, z \mid z \in \mathbb{C},\ |z| = 1\}$
- $\mathbb{D}_+ := \{\, z \mid z \in \mathbb{C},\ |z| > 1\}$



Definition

(i) $\mathbf{H}_\infty(\mathbb{D}_+) := \left\{ \hat{T} \;\middle|\; \sup_{z\in\mathbb{D}_+} \sigma_{\max}\left(\hat{T}[z]\right) < \infty \right\}$

(ii) Given $\hat{T} \in \mathbf{H}_\infty$,

$$\|\hat{T}\|_\infty := \sup_{z\in\mathbb{D}_+} \sigma_{\max}\left(\hat{T}[z]\right)$$

$\mathbf{H}_\infty$ Norm (2/2)

Property $\hat{T}[z] = C(zI - A)^{-1}B + D$

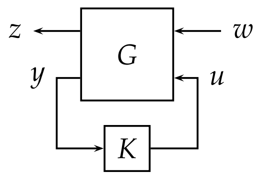
(i) $\mathrm{eig}(A) \subset \mathbb{D} \;\Rightarrow\; \hat{T} \in \mathbf{H}_\infty$

(ii) $\hat{T} \in \mathbf{H}_\infty \;\Rightarrow\; \mathrm{eig}(A) \subset \mathbb{D}$ if stabilizable and detectable

(iii) Given $T \in \mathbf{H}_\infty$,

$$\begin{aligned}\|\hat{T}\|_\infty &= \sup_{z\in\partial\mathbb{D}} \sigma_{\max}\left(\hat{T}[z]\right)\\ &= \sup_{\theta\in[0,2\pi)} \sigma_{\max}\left(\hat{T}[e^{j\theta}]\right)\end{aligned}$$

Design Framework



- Given: Generalized plant G

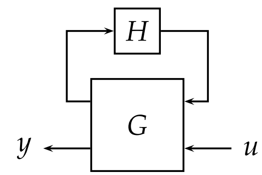
$$\hat{G}[z] = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (zI - A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{bmatrix}$$

- Find: Controller K

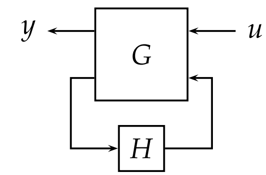
$$\hat{K}[z] = C_K(zI - A_K)^{-1}B_K + D_K$$

- $z[z] = \mathcal{F}_\ell\left(\hat{G}, \hat{K}\right)[z]\; w[z]$

LFT (Linear Fractional Transformation)



$\mathcal{F}_u(G, H)$: upper LFT



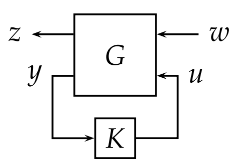
$\mathcal{F}_\ell(G, H)$: lower LFT

$$\begin{aligned} \mathcal{F}_u(G, H) &:= G_{22} + G_{21}H(I - G_{11}H)^{-1}G_{12} \\ \mathcal{F}_\ell(G, H) &:= G_{11} + G_{12}H(I - G_{22}H)^{-1}G_{21} \end{aligned}$$

Example

$$C(zI - A)^{-1}B + D = \mathcal{F}_u\left(M, z^{-1}I\right), \quad M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$\mathbf{H}_\infty$ Control Synthesis



$$\mathcal{F}_\ell\left(\hat{G}, \hat{K}\right)[z] = C_{c\ell}(zI - A_{c\ell})^{-1}B_{c\ell} + D_{c\ell}$$

$$\begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix} := \left[\begin{array}{cc|c} A & 0 & B_1 \\ 0 & 0 & 0 \\ \hline C_1 & 0 & D_{11} \end{array}\right] + \left[\begin{array}{cc} B_2 & 0 \\ 0 & I \\ \hline D_{12} & 0 \end{array}\right] \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \left[\begin{array}{cc|c} C_2 & 0 & D_{21} \\ 0 & I & 0 \end{array}\right]$$

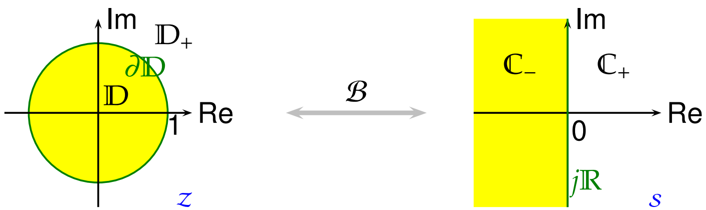
$\mathbf{H}_\infty$ control synthesis Given G. Find K such that

- $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D}$
- $\left\| \mathcal{F}_\ell\left(\hat{G}, \hat{K}\right) \right\|_\infty < 1$

Outline

Bilinear Transformation (1/2)

Bilinear Transformation $z = \dfrac{1+s}{1-s}$



Property: Given $\hat{G}$. Let $\hat{G}_c(s) := \hat{G}\left[\dfrac{1+s}{1-s}\right]$

- $\hat{G} \in \mathbf{H}_\infty(\mathbb{D}_+) \Leftrightarrow \hat{G}_c \in \mathbf{H}_\infty(\mathbb{C}_+)$
- Suppose $\hat{G} \in \mathbf{H}_\infty(\mathbb{D}_+)$. $\|\hat{G}\|_\infty = \|\hat{G}_c\|_\infty$

Bilinear Transformation (2/2)

Given $\hat{G}[z] = C(zI - A)^{-1}B + D$ with $-1 \notin \mathrm{eig}(A)$

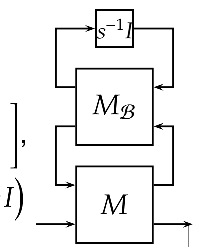
Fact: $\hat{G}_c(s) := \hat{G}\left[\dfrac{1+s}{1-s}\right] = C_c(sI - A_c)^{-1}B_c + D_c$

$$\begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} := \begin{bmatrix} (A-I)(A+I)^{-1} & \sqrt{2}(A+I)^{-1}B \\ \sqrt{2}C(A+I)^{-1} & \hat{G}[-1] \end{bmatrix}$$

Proof: Notice

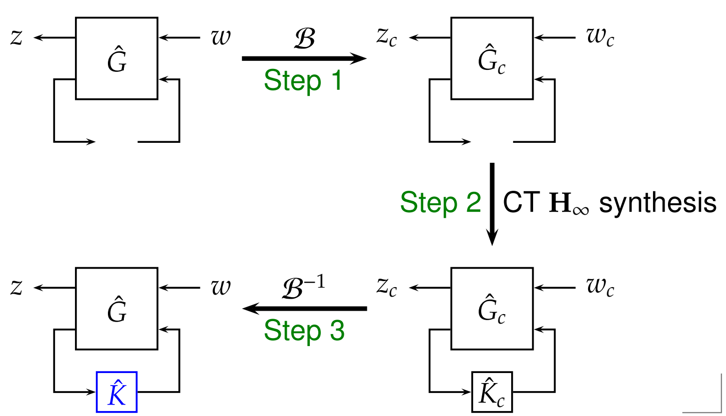
- $\hat{G}[z] = \mathcal{F}_u\left(M, z^{-1}I\right), \quad M := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$
- $z^{-1}I = \dfrac{1-s}{1+s}I = \mathcal{F}_u\left(M_{\mathcal{B}}, s^{-1}I\right), \quad M_{\mathcal{B}} := \begin{bmatrix} -I & \sqrt{2}I \\ \sqrt{2}I & -I \end{bmatrix},$

$\hat{G}_c(s) = \mathcal{F}_u\left(M, \mathcal{F}_u\left(M_{\mathcal{B}}, s^{-1}I\right)\right) = \mathcal{F}_u\left(M_{\mathcal{B}} \star M, s^{-1}I\right)$



Reduction to Continuous-Time $\mathbf{H}_\infty$ Control Synthesis

Design Procedure



Outline

Bounded Real Lemma

Notation

- $A > 0$: $A \in \mathbb{R}^{n\times n}$ is positive definite
- $A > B$: $A - B > 0$

Lemma Given $\hat{T}[z] = C(zI - A)^{-1}B + D$, $\gamma > 0$

(i) $\mathrm{eig}(A) \subset \mathbb{D}$ and $\|\hat{T}\|_\infty < \gamma$

$\Leftrightarrow$ (ii) $\exists P = P^\mathsf{T} > 0$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} P & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0$$

- Analysis via convex optimization

Schur Complement

Lemma SAE

- $\begin{bmatrix} M_1 & M_3 \\ M_3^\mathsf{T} & M_2 \end{bmatrix} > 0$
- $M_1 > 0$ and $M_2 - M_3^\mathsf{T} M_2^{-1} M_3 > 0$
- $M_2 > 0$ and $M_1 - M_3 M_2^{-1} M_3^\mathsf{T} > 0$

Proof: easy

Elimination Lemma

Notation: Given $A \in \mathbb{R}^{n\times m}$, $\mathrm{rank}(A) = r$
$A^\perp \in \mathbb{R}^{(n-r)\times n}$ is any matrix satisfying

$$A^\perp A = 0, \quad A^\perp A^{\perp\mathsf{T}} > 0$$

Lemma Given $L \in \mathbb{R}^{n\times m}$, $R \in \mathbb{R}^{p\times n}$, and $M = M^\mathsf{T} \in \mathbb{R}^{n\times n}$

(i) $\exists Z \in \mathbb{R}^{m\times p}$, $LZR + (LZR)^\mathsf{T} + M < 0$

$\Leftrightarrow$ (ii)

$$\begin{cases} L^\perp M L^{\perp\mathsf{T}} < 0 & \text{or} \quad LL^\mathsf{T} > 0 \\ R^{\mathsf{T}\perp} M R^{\mathsf{T}\perp\mathsf{T}} < 0 & \text{or} \quad R^\mathsf{T} R > 0 \end{cases}$$

LMI Feasibility Condition

Theorem Given G. SAE

(i) $\exists$ a solution to $\mathbf{H}_\infty$ control synthesis problem

(ii) $\exists\, X = X^\mathsf{T} > 0$, $Y = Y^\mathsf{T} > 0$,

$$\begin{cases} \mathcal{B}^\perp \left(\mathcal{A} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \mathcal{A}^\mathsf{T} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \right) \mathcal{B}^{\perp\mathsf{T}} & < & 0 \\ \mathcal{C}^{\mathsf{T}\perp} \left(\mathcal{A}^\mathsf{T} \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \mathcal{A} - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \right) \mathcal{C}^{\mathsf{T}\perp\mathsf{T}} & < & 0 \\ \begin{bmatrix} X & I \\ I & Y \end{bmatrix} & \geq & 0 \end{cases}$$

where $\mathcal{A} := \begin{bmatrix} A & B_1 \\ C_1 & D_{11} \end{bmatrix}$, $\mathcal{B} := \begin{bmatrix} B_2 \\ D_{12} \end{bmatrix}$, $\mathcal{C} := \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$

Sketch of Proof (1/3)

- Step 1: Bounded Real Lemma

$$\begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix}^{\mathsf T} \begin{bmatrix} P_{c\ell} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix} - \begin{bmatrix} P_{c\ell} & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} \begin{bmatrix} P_{c\ell} & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix}^{\mathsf T} \\ \begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix} & \begin{bmatrix} P_{c\ell}^{-1} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix} > 0$$

- Step 2: Notice that

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & I \end{bmatrix}$$

Sketch of Proof (2/3)

$$\begin{bmatrix} \begin{bmatrix} Y & 0 & P_3 \\ 0 & I & 0 \\ P_3^{\mathsf T} & 0 & P_2 \end{bmatrix} & \begin{bmatrix} \mathcal{A}^{\mathsf T} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{C}^{\mathsf T} & 0 \\ 0 & I \end{bmatrix} Z_K^{\mathsf T} \begin{bmatrix} \mathcal{B}^{\mathsf T} & 0 \\ 0 & I \end{bmatrix} \\ \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{B} & 0 \\ 0 & I \end{bmatrix} Z_K \begin{bmatrix} \mathcal{C} & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} X & 0 & Q_3 \\ 0 & I & 0 \\ Q_3^{\mathsf T} & 0 & Q_2 \end{bmatrix} \end{bmatrix} > 0$$

where $P_{c\ell} =: \begin{bmatrix} Y & P_3 \\ P_3^{\mathsf T} & P_2 \end{bmatrix}$, $P_{c\ell}^{-1} =: \begin{bmatrix} X & Q_3 \\ Q_3^{\mathsf T} & Q_2 \end{bmatrix}$, $Z_K := \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}$

$$Y = (X - Q_3 Q_2^{-1} Q_3^{\mathsf T})^{-1}$$

$$\Rightarrow \quad X - Y^{-1} = Q_3 Q_2^{-1} Q_3^{\mathsf T} \geq 0$$

$$\Rightarrow \quad \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0$$

Sketch of Proof (3/3)

$$\mathcal{L} Z_K \mathcal{R} + (\mathcal{L} Z_K \mathcal{R})^{\mathsf T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$$

$$\mathcal{L} := \begin{bmatrix} 0 \\ \begin{bmatrix} \mathcal{B} & 0 \\ 0 & I \end{bmatrix} \end{bmatrix}, \quad \mathcal{R} := \begin{bmatrix} \begin{bmatrix} \mathcal{C} & 0 \\ 0 & I \end{bmatrix} & 0 \end{bmatrix}, \quad \mathcal{M}(\mathcal{P}_{c\ell}) := \begin{bmatrix} \begin{bmatrix} Y & 0 & P_3 \\ 0 & I & 0 \\ P_3^{\mathsf T} & 0 & P_2 \end{bmatrix} & \begin{bmatrix} \mathcal{A}^{\mathsf T} & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} X & 0 & Q_3 \\ 0 & I & 0 \\ Q_3^{\mathsf T} & 0 & Q_2 \end{bmatrix} \end{bmatrix}$$

- Step 3: Elimination Lemma for Z_K with

$$\mathcal{L}^{\perp} = \begin{bmatrix} I & 0 \\ 0 & \begin{bmatrix} \mathcal{B}^{\perp} & 0 \end{bmatrix} \end{bmatrix}, \quad \mathcal{R}^{\mathsf T\perp} = \begin{bmatrix} \begin{bmatrix} \mathcal{C}^{\mathsf T\perp} & 0 \end{bmatrix} & 0 \\ 0 & I \end{bmatrix}$$

Outline

Recovering $P_{c\ell}$

- Solution to synthesis LMI (X, Y) satisfying $\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0$
  - otherwise replace by $(X + \varepsilon I,\ Y + \varepsilon I)$
  - $(X + \varepsilon I,\ Y + \varepsilon I)$ solves synthesis LMI for small $\varepsilon > 0$
- Fact: $\exists P_2, Q_2$ such that

$$\begin{bmatrix} Y & N \\ N^{\mathsf T} & P_2 \end{bmatrix} > 0, \quad \begin{bmatrix} X & M \\ M^{\mathsf T} & Q_2 \end{bmatrix} > 0, \quad \underbrace{\begin{bmatrix} Y & N \\ N^{\mathsf T} & P_2 \end{bmatrix}}_{P_{c\ell}} \underbrace{\begin{bmatrix} X & M \\ M^{\mathsf T} & Q_2 \end{bmatrix}}_{P_{c\ell}^{-1}} = I$$

where $MN^{\mathsf T} := I - XY$

Note: M, N are

- nonsingular
- computable via SVD

Controller Construction

- Naive Algorithm:
  - Substitute $P_{c\ell}$ back to $\mathcal{L} Z_K \mathcal{R} + (\mathcal{L} Z_K \mathcal{R})^{\mathsf T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$
  - Solve as an LMI in Z_K (feasible !)
  - Works well, but numerically expensive
- Sofisticated algorithm:
  - Without recovering $P_{c\ell}$ explicitly
  - Decomposing $\mathcal{L} Z_K \mathcal{R} + (\mathcal{L} Z_K \mathcal{R})^{\mathsf T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$ by using
    - $P_{c\ell} \begin{bmatrix} X & I \\ M^{\mathsf T} & 0 \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & N^{\mathsf T} \end{bmatrix}$
    - $\begin{bmatrix} X & I \\ M^{\mathsf T} & 0 \end{bmatrix}^{\mathsf T} P_{c\ell} \begin{bmatrix} X & I \\ M^{\mathsf T} & 0 \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$
    - $\begin{bmatrix} I & Y \\ 0 & N^{\mathsf T} \end{bmatrix}^{\mathsf T} P_{c\ell}^{-1} \begin{bmatrix} I & Y \\ 0 & N^{\mathsf T} \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$

Decomposing $\mathcal{L} Z_K \mathcal{R} + (\mathcal{L} Z_K \mathcal{R})^{\mathsf T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$ (1/3)

Lemma SAE

(i) $\exists Z_K$ such that $\mathcal{L} Z_K \mathcal{R} + (\mathcal{L} Z_K \mathcal{R})^{\mathsf T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$

(ii) $\exists (\tilde B_K, \tilde C_K, D_K)$ such that

$$\begin{bmatrix} X & L_X(\tilde C_K) \\ L_X^{\mathsf T}(\tilde C_K) & \Delta(D_K) \end{bmatrix} > 0, \quad \begin{bmatrix} Y & L_Y(\tilde B_K) \\ L_Y^{\mathsf T}(\tilde B_K) & \Delta(D_K) \end{bmatrix} > 0$$

where

$$\begin{bmatrix} L_X(\tilde C_K) \\ L_Y(\tilde B_K) \\ \Delta(D_K) \end{bmatrix} := \left[\begin{array}{cc} \begin{bmatrix} I & 0 \end{bmatrix} & \tilde C_K^{\mathsf T} \mathcal{B}^{\mathsf T} + \begin{bmatrix} X & 0 \end{bmatrix} \mathcal{A}^{\mathsf T} \\ \hline \tilde B_K \mathcal{C} + \begin{bmatrix} Y & 0 \end{bmatrix} \mathcal{A} & \begin{bmatrix} I & 0 \end{bmatrix} \\ \hline \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} & (\mathcal{A} + \mathcal{B} D_K \mathcal{C})^{\mathsf T} \\ \mathcal{A} + \mathcal{B} D_K \mathcal{C} & \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \end{array}\right]$$

Decomposing $\mathcal{L} Z_K \mathcal{R} + (\mathcal{L} Z_K \mathcal{R})^{\mathsf T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$ (2/3)

Sketch of Proof:

(i) $\Leftrightarrow \exists (\tilde A_K, \tilde B_K, \tilde C_K, D_K)$ such that

$$\begin{bmatrix} \begin{bmatrix} X & \tilde A_K^{\mathsf T} \\ \tilde A_K & Y \end{bmatrix} & \begin{bmatrix} L_X(\tilde C_K) \\ L_Y(\tilde B_K) \end{bmatrix} \\ \begin{bmatrix} L_X^{\mathsf T}(\tilde C_K) & L_Y^{\mathsf T}(\tilde B_K) \end{bmatrix} & \Delta(D_K) \end{bmatrix} > 0$$

by taking

$$\begin{bmatrix} \tilde A_K & \tilde B_K \\ \tilde C_K & D_K \end{bmatrix} = \begin{bmatrix} Y & N & 0 \\ 0 & 0 & I \end{bmatrix} \left(\begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & I \\ I & 0 \end{bmatrix} Z_K \begin{bmatrix} C_2 & 0 & I \\ 0 & I & 0 \end{bmatrix} \right) \begin{bmatrix} X & 0 \\ M^{\mathsf T} & 0 \\ 0 & I \end{bmatrix}$$

$\Leftrightarrow$ (ii): Set $\tilde A_K = L_Y(\tilde B_K) \Delta^{-1}(D_K) L_X^{\mathsf T}(\tilde C_K)$

Decomposing $\mathcal{L}Z_K\mathcal{R} + (\mathcal{L}Z_K\mathcal{R})^\mathsf{T} + \mathcal{M}(\mathcal{P}_{c\ell}) > 0$ (3/3)

Given solution (X, Y) to synthesis LMI

Lemma $^\forall D_K$ such that $\Delta(D_K) > 0$

- $^\exists \tilde{B}_K$ such that $\begin{bmatrix} Y & L_Y(\tilde{B}_K) \\ L_Y^\mathsf{T}(\tilde{B}_K) & \Delta(D_K) \end{bmatrix} > 0$
- $^\exists \tilde{C}_K$ such that $\begin{bmatrix} X & L_X(\tilde{C}_K) \\ L_X^\mathsf{T}(\tilde{C}_K) & \Delta(D_K) \end{bmatrix} > 0$

Sketch of proof: Invoking Elimination Lemma for $\tilde{B}_K$,

$$\begin{bmatrix} Y & L_Y(\tilde{B}_K) \\ L_Y^\mathsf{T}(\tilde{B}_K) & \Delta(D_K) \end{bmatrix} > 0 \Leftrightarrow \begin{cases} \Delta(D_K) > 0 \\ C^{\mathsf{T}\perp}\left(\mathcal{A}^\mathsf{T}\begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}\mathcal{A} - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix}\right)C^{\mathsf{T}\perp\mathsf{T}} < 0 \end{cases}$$

Controller Construction Algorithm

Algorithm

- Step 0: (X, Y): solution to synthesis LMI
- Step 1: Find D_K such that $\Delta(D_K) > 0$
- Step 2: Find $\tilde{B}_K$ and $\tilde{C}_K$ such that

  $$\begin{bmatrix} X & L_X(\tilde{C}_K) \\ L_X^\mathsf{T}(\tilde{C}_K) & \Delta(D_K) \end{bmatrix} > 0, \quad \begin{bmatrix} Y & L_Y(\tilde{B}_K) \\ L_Y^\mathsf{T}(\tilde{B}_K) & \Delta(D_K) \end{bmatrix} > 0$$

- Step 3: Set $\tilde{A}_K = L_Y(\tilde{B}_K)\Delta^{-1}(D_K)L_X^\mathsf{T}(\tilde{C}_K)$
- Step 4: Recover Z_K by

  $$Z_K = \begin{bmatrix} I & 0 \\ 0 & N^{-1} \end{bmatrix}\left(\begin{bmatrix} I & 0 \\ -YB_2 & I \end{bmatrix}\begin{bmatrix} D_K & \tilde{C}_K \\ \tilde{B}_K & \tilde{A}_K \end{bmatrix}\begin{bmatrix} I & -C_2X \\ 0 & I \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & YAX \end{bmatrix}\right)\begin{bmatrix} I & 0 \\ 0 & M^{-\mathsf{T}} \end{bmatrix}$$

♣ SVD-based methods available for Step 1 and 2

References

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