Sampled-Data LQ Optimal Control

Lecture #4

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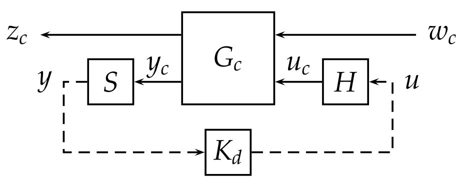
Outline

- Stability of Sampled-Data Systems
  - Internal Stability
  - Properties of Discretized Systems
  - Input-Output Stability
- Sampled-Data LQ Optimal Control
  - Problem Formulation
  - Reduction to Discrete-Time Problem

Sampled-Data Systems



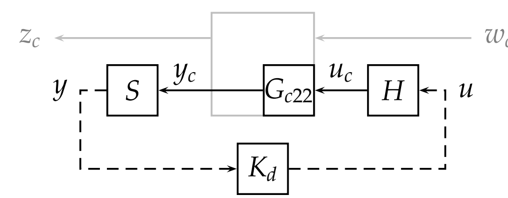
$$G_c : \begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_{c1} & B_{c2} \\ C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ w_c(t) \\ u_c(t) \end{bmatrix}$$

$$K_d : \begin{bmatrix} x_K[k+1] \\ u[k] \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K[k] \\ y[k] \end{bmatrix}$$

$$S : y[k] = y_c(kh)$$

$$H : u_c(kh+\theta) = u[k], \quad {}^\forall \theta \in [0, h)$$

Internal Stability of Sampled-Data Systems (1/3)

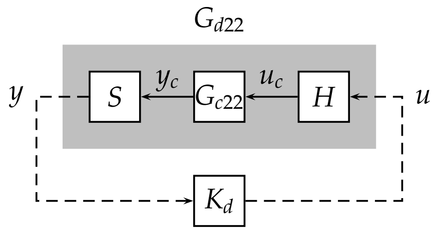


Definition Fix $w_c = 0$

SDS is internally stable

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} x_c(t) \to 0 & (t \to \infty) \\ x_K[k] \to 0 & (k \to \infty) \end{cases}, \quad {}^\forall x_c(0),\ x_K[0]$$

Internal Stability of Sampled-Data Systems (2/3)



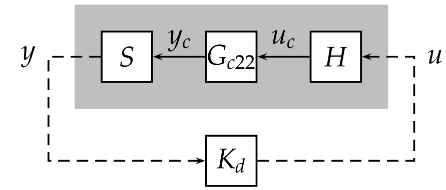
State-space model of G_{d22}

$$\begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B_2 \\ C_2 & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} := \begin{bmatrix} e^{A_c h} & \int_0^h e^{A_c \eta} B_{c2}\, d\eta \\ C_{c2} & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

Sketch of Proof: Define $x[k] := x_c(kh)$

$$x[k+1] = e^{A_c h} x[k] + \int_{kh}^{(k+1)h} e^{A_c((k+1)h-\eta)} B_{c2} u_c(\eta)\, d\eta$$

Internal Stability of Sampled-Data Systems (3/3)



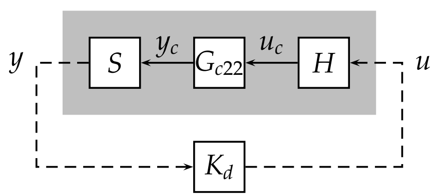
$$\begin{bmatrix} x[k+1] \\ x_K[k+1] \end{bmatrix} = A_{c\ell} \begin{bmatrix} x[k] \\ x_K[k] \end{bmatrix}; \quad A_{c\ell} := \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}$$

Theorem SAE

- SDS is internally stable
- $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D}$

Sketch of Proof: $x_c(kh+\theta) = e^{A_c \theta} x[k] + \int_0^\theta e^{A_c(\theta-\eta)} B_{c2}\, d\eta\, (C_K x_K[k] + D_K C_2 x[k])$

Properties of Discretized Systems (1/3)



Fact: $^\exists K_d$ such that $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D}$ iff

- (A, B_2): stabilizable
- (A, C_2): detectable

Claim:

- (A_c, B_{c2}): stabilizable $\not\Rightarrow$ (A, B_2): stabilizable
- (A_c, C_{c2}): detectable $\not\Rightarrow$ (A, C_2): detectable

Properties of Discretized Systems (2/3)

Counter Example:

- $\lambda_1, \lambda_2 \in \mathrm{eig}(A_c) \;\Rightarrow\; e^{\lambda_1 h}, e^{\lambda_2 h} \in \mathrm{eig}(A)$
- $\lambda_1 \neq \lambda_2 \quad \not\Rightarrow \quad e^{\lambda_1 h} \neq e^{\lambda_2 h}$
- $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2)$, $\mathrm{Im}(\lambda_1) - \mathrm{Im}(\lambda_2) = \frac{2\pi}{h} j \cdot n \;\Rightarrow\; e^{\lambda_1 h} = e^{\lambda_2 h}$

$$\begin{bmatrix} A_c & B_{c2} \\ C_{c2} & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right]$$: stabilizable, detectable

- $\mathrm{eig}(A_c) = \{j, -j\}$
- $h = n\pi \;\rightarrow\; \mathrm{rank} \begin{bmatrix} zI - A \\ C_2 \end{bmatrix} = \mathrm{rank} \begin{bmatrix} z+1 & 0 \\ 0 & z+1 \\ 0 & 1 \end{bmatrix}$

Properties of Discretized Systems (3/3)

Definition Given A_c, $h > 0$

h is pathological if $^{\exists}\lambda_1, \lambda_2 \in \mathrm{eig}(A_c)$, $k \in \mathbb{Z}$ such that

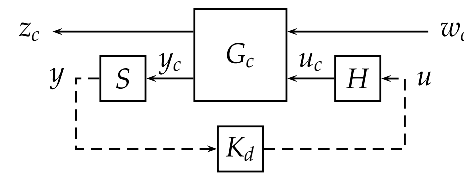
$$\lambda_1 \neq \lambda_2, \quad \lambda_1 - \lambda_2 \in j\mathbb{R}, \quad |\lambda_1 - \lambda_2| = \omega_s k, \quad \omega_s := \frac{2\pi}{h}$$

sampling frequency

Theorem Suppose h is not pathological

- (A_c, B_{c2}): stabilizable $\Leftrightarrow$ (A, B_2): stabilizable
- (A_c, C_{c2}): detectable $\Leftrightarrow$ (A, C_2): detectable

$\mathbf{L}_2$-Stability of Sampled-Data Systems (1/3)



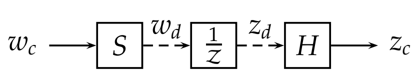
Definition SDS is $\mathbf{L}_p$-stable if $^{\forall}w_c \in \mathbf{L}_p \Rightarrow z_c \in \mathbf{L}_p$

Claim:

- Internal stability $\Rightarrow$ $\mathbf{L}_\infty$-stability
- Internal stability $\not\Rightarrow$ $\mathbf{L}_2$-stability

$\mathbf{L}_2$-Stability of Sampled-Data Systems (2/3)

Counter Example:

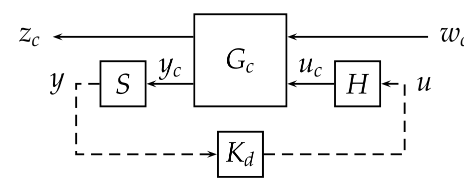


- Internally stable ($A_{c\ell} = 0$)
- Set w_c by $w_c(kh + \theta) := \begin{cases} 1, & \theta \in [0, h/(k+1)^2) \\ 0, & \theta \in [h/(k+1)^2, h) \end{cases}$

$$\int_0^\infty w_c^2(t)\,\mathrm{d}t = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} < \infty \quad \Rightarrow \quad w_c \in \mathbf{L}_2$$

$$z_c(t) \equiv 1, \ ^{\forall}t \geq h \quad \Rightarrow \quad z_c \notin \mathbf{L}_2$$

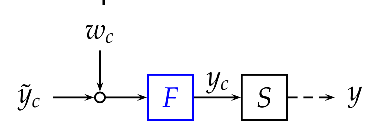
$\mathbf{L}_2$-Stability of Sampled-Data Systems (3/3)



Theorem Suppose $D_{c21} = 0$

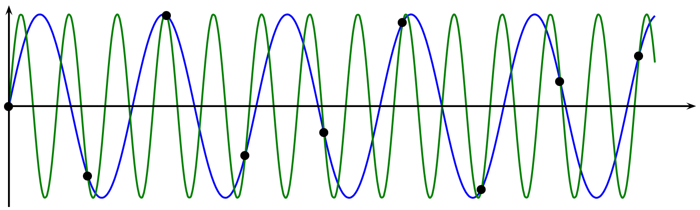
Internal stability $\Rightarrow$ $\mathbf{L}_2$-stability

Message: Put low-pass filter F before sampler S

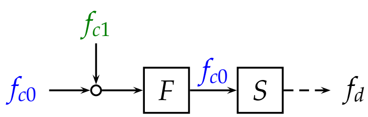


Anti-Aliasing Filter

Aliasing



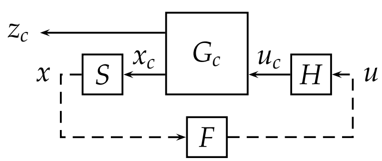
- Sampler can produce same discrete-time signal from distinct continuous-time signals
- Usually we put low-pass filter before sampler to avoid it



Anti-Aliasing Filter

Outline

Sampled-Data LQ Optimal Control Problem



Given:

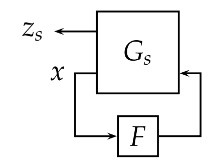
$$\begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(t) \\ u_c(t) \end{bmatrix}, \quad x_c(0) = x_0$$

Problem Find $F \in \mathbb{R}^{m_2 \times n}$ such that

- $\mathcal{F}_\ell(G_c, HFS)$ is internally stable
- minimize $\|z_c\|_2$

Reduction to Discrete-Time Problem (1/2)

Theorem



- $\mathcal{F}_\ell(G_c, HFS)$ is internally stable
  $\Leftrightarrow$ $\mathcal{F}_\ell(G_s, F)$ is internally stable
- Supposing stability $\|z_c\|_2 = \|z_s\|_2$

where

$$\begin{bmatrix} x[k+1] \\ z_s[k] \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}, \quad x[0] = x_0$$

(C, D) is any pair of matrices satisfying

$$\begin{bmatrix} C^{\mathrm{T}} \\ D^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} = \int_0^h \exp\left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} t \right) \begin{bmatrix} C_c^{\mathrm{T}} \\ D_c^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} C_c & D_c \end{bmatrix} \exp\left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix} t \right) \mathrm{d}t$$

and $A := \mathrm{e}^{A_c h}, \quad B := \int_0^h \mathrm{e}^{A_c t} B_c \,\mathrm{d}t$

Reduction to Discrete-Time Problem (2/2)

Design Procedure:

- Step 1: Reduction to Discrete-Time Problem
  Compute A, B, C, D
- Step 2: Discrete-Time LQ Optimal Control Problem
  Find optimal gain F for (A, B, C, D)

Proof of Theorem:

$$z(kh+\theta) = C_c e^{A_c\theta} x_c(kh) + D_c u[k] = \begin{bmatrix} C_c & D_c \end{bmatrix} \exp\left(\begin{bmatrix} A_c & B_c \\ 0 & 0 \end{bmatrix} \theta \right) \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

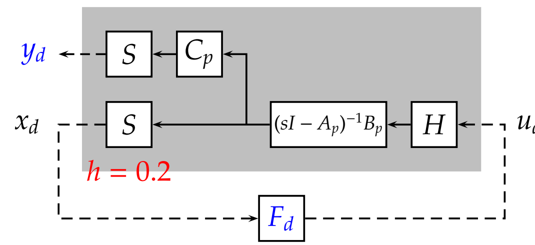
for $\theta \in [0, h)$

Numerical Example (1/4)

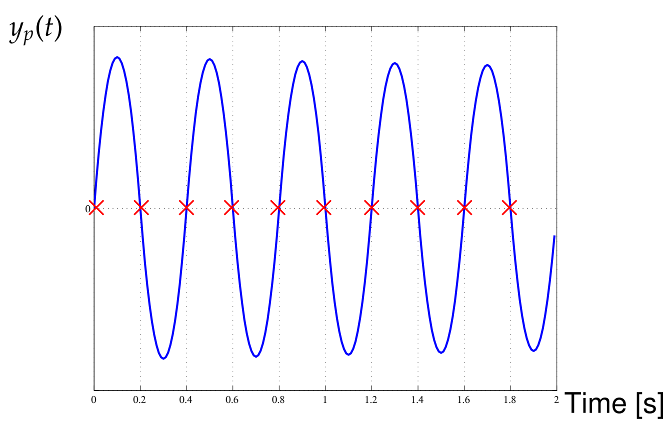
Plant:

$$\hat{P}(s) = \frac{1}{s^2 + 0.1s + 1}; \quad \begin{bmatrix} \dot{x}_p(t) \\ y_p(t) \end{bmatrix} = \left[\begin{array}{cc|c} -0.1 & 1 & 1 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array} \right] \begin{bmatrix} x_p(t) \\ u_p(t) \end{bmatrix}, \quad x_p(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Discrete-Time Design: Find F_d which minimizes $\|y_d\|_2$



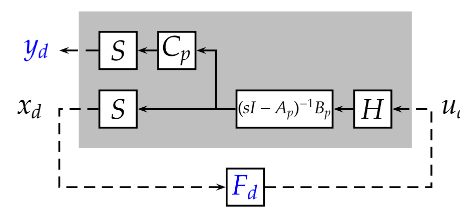
Numerical Example (2/4)



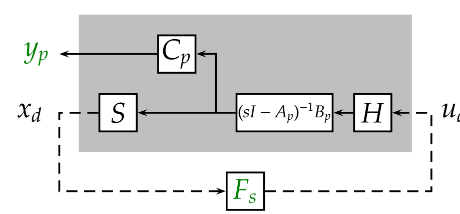
- Deadbeat at sampling instants
- Oscillating intersample behavior

Numerical Example (3/4)

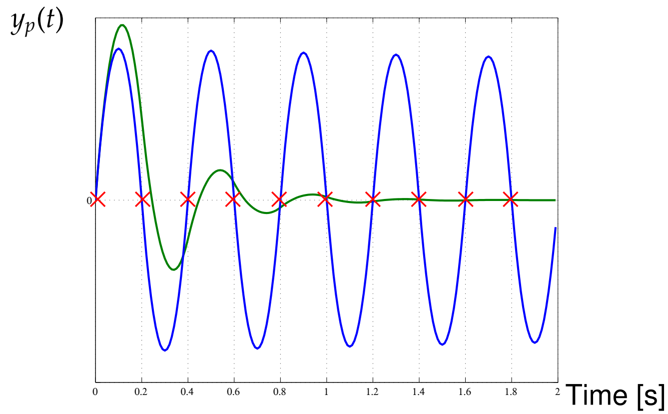
Discrete-Time Design: Find F_d which minimizes $\|y_d\|_2$



Sampled-Data Design: Find F_s which minimizes $\|y_p\|_2$



Numerical Example (4/4)



- Less-oscillating response by Sampled-data design
- Discrete design is better at sampling instants