Time-Invariant Model of Sampled-Data Systems

Lecture #4

Hisaya Fujioka
fujioka@{i.kyoto-u.ac.jp,math.kth.se}

Graduate School of Informatics, Kyoto University

Outline

- Motivating Example: Discrete-Time Periodic System
- Lifting Signals
- Lifting Sampled-Data Systems
- Transfer Function of Sampled-Data Systems
  - Frequency Response of Sampled-Data Systems
  - $\mathbf{H}_\infty$-Norm of Sampled-Data Systems

TI Model for Discrete-Time Periodic System (1/2)

Motivating Example: Discrete-time 2-periodic system

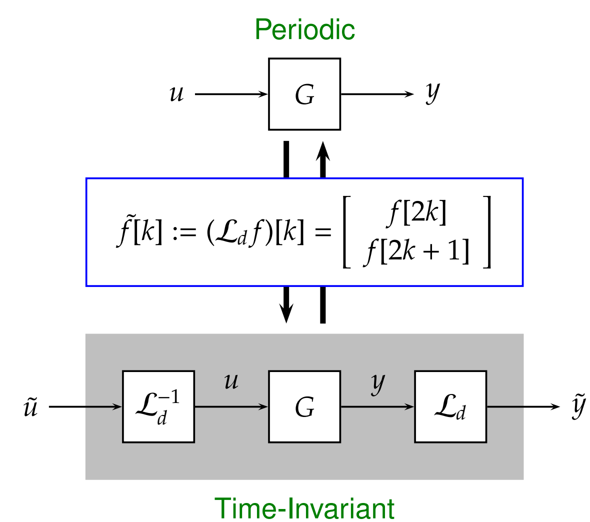
$$G : \begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{cases} \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} & (k : \text{even}) \\ \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} & (k : \text{odd}) \end{cases}$$

Alternative representation:

$$\begin{bmatrix} x[2k+2] \\ \hline y[2k] \\ y[2k+1] \end{bmatrix} = \left[\begin{array}{c|cc} A_1A_0 & A_1B_0 & B_1 \\ \hline C_0 & D_0 & 0 \\ C_1A_0 & C_1B_0 & D_1 \end{array}\right] \begin{bmatrix} x[2k] \\ \hline u[2k] \\ u[2k+1] \end{bmatrix}$$

Time-Invariant

TI Model for Discrete-Time Periodic System (2/2)



Outline

- Motivating Example: Discrete-Time Periodic System
- Lifting Signals
- Lifting Sampled-Data Systems
- Transfer Function of Sampled-Data Systems
  - Frequency Response of Sampled-Data Systems
  - $\mathbf{H}_\infty$-Norm of Sampled-Data Systems

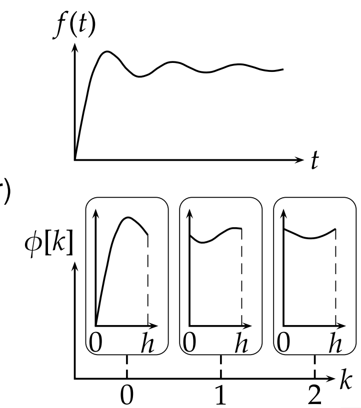
Lifting Continuous-Time Signals (1/2)

Lifting: function $\rightarrow$ sequence taking values in function space

function: $f \in \mathbf{L}_{2e}[0, \infty)$

Lifting $\mathcal{L}$ $\quad h > 0$ (parameter)

sequence:
$\phi \colon \mathbb{Z}_+ \rightarrow \mathbf{L}_2[0, h]$
$\phi[k](\theta) = f(kh + \theta)$



Lifting Continuous-Time Signals (2/2)

Theorem

- $f \in \mathbf{L}_2[0, \infty) \;\Leftrightarrow\; \mathcal{L}f \in \ell_2(\mathbb{Z}_+, \mathbf{L}_2[0, h])$
- $\|f\|_2 = \|\mathcal{L}f\|_2$ if either $f \in \mathbf{L}_2$ or $\mathcal{L}f \in \ell_2$ holds

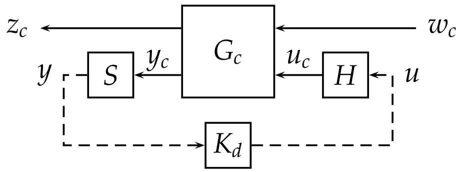
Proof:

$$\sum_{k=0}^{\infty} \|(\mathcal{L}f)[k]\|_2^2 = \sum_{k=0}^{\infty} \int_0^h f^*(kh+\theta) f(kh+\theta)\, d\theta = \int_0^\infty f^*(t) f(t)\, dt$$

Outline

- Motivating Example: Discrete-Time Periodic System
- Lifting Signals
- Lifting Sampled-Data Systems
- Transfer Function of Sampled-Data Systems
  - Frequency Response of Sampled-Data Systems
  - $\mathbf{H}_\infty$-Norm of Sampled-Data Systems

Lifting Sampled-Data Systems (1/6): Setup



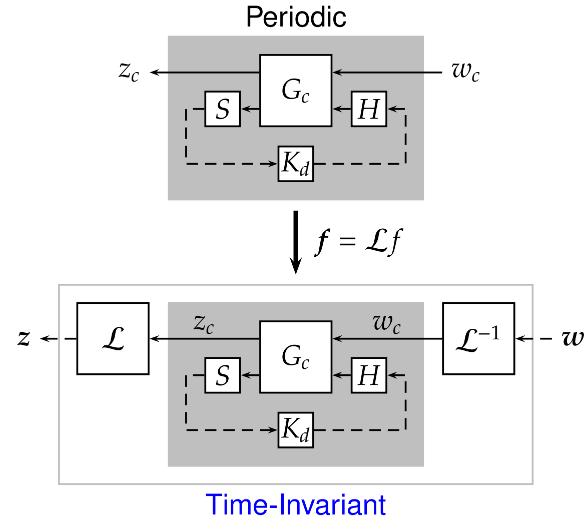
$$G_c : \begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_{c1} & B_{c2} \\ C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ w_c(t) \\ u_c(t) \end{bmatrix}$$

$$K_d : \begin{bmatrix} x_K[k+1] \\ u[k] \end{bmatrix} = \begin{bmatrix} A_K & B_K \\ C_K & D_K \end{bmatrix} \begin{bmatrix} x_K[k] \\ y[k] \end{bmatrix}$$

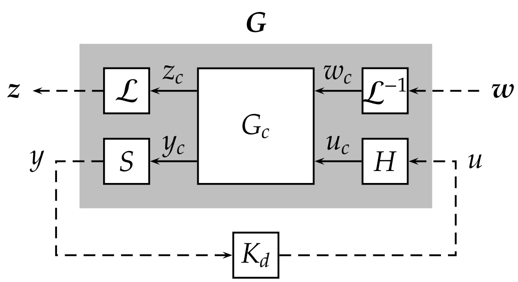
$$S : y[k] = y_c(kh)$$

$$H : u_c(kh + \theta) = u[k], \quad {}^\forall \theta \in [0, h)$$

Lifting Sampled-Data Systems (2/6)



Lifting Sampled-Data Systems (3/6)



Claim G is time-invariant

$$G : \begin{bmatrix} x[k+1] \\ \boldsymbol{z}[k] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & \boldsymbol{B}_1 & B_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \\ C_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x[k] \\ \boldsymbol{w}[k] \\ u[k] \end{bmatrix}$$

$$x[k] := x_c(kh)$$

Lifting Sampled-Data Systems (4/6)

$$A := \mathrm{e}^{A_c h}, \quad B_2 := \int_0^h \mathrm{e}^{A_c \xi} B_{c2}\, d\xi, \quad C_2 := C_{c2}$$

$$\boldsymbol{B}_1 : \mathbf{L}_2([0,h], \mathbb{R}^{m_1}) \to \mathbb{R}^n, \quad \boldsymbol{B}_1 w := \int_0^h \mathrm{e}^{A_c(h-\xi)} B_{c1} w(\xi)\, d\xi$$

$$\boldsymbol{C}_1 : \mathbb{R}^n \to \mathbf{L}_2([0,h], \mathbb{R}^{p_1}), \quad \boldsymbol{D}_{12} : \mathbb{R}^{m_2} \to \mathbf{L}_2([0,h], \mathbb{R}^{p_1}),$$

$$\left(\begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix} v \right)(\theta) := \begin{bmatrix} C_{c1} & D_{c12} \end{bmatrix} \exp\left(\begin{bmatrix} A_c & B_{c2} \\ 0 & 0 \end{bmatrix} \theta \right)$$

$$\boldsymbol{D}_{11} : \mathbf{L}_2([0,h], \mathbb{R}^{m_1}) \to \mathbf{L}_2([0,h], \mathbb{R}^{p_1}),$$

$$(\boldsymbol{D}_{11} w)(\theta) = C_{c1} \int_0^\theta \mathrm{e}^{A_c(\theta-\xi)} B_{c1} w(\xi)\, d\xi + D_{c11} w(\theta)$$

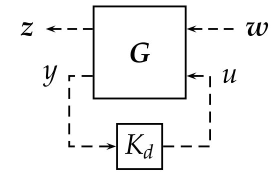
Lifting Sampled-Data Systems (5/6)

Proof:

$$\begin{aligned} x[k+1] &= e^{A_c h}x[k] + \int_0^h e^{A_c(h-\xi)}B_{c1}w_c(kh+\xi)\,d\xi + \int_0^h e^{A_c\xi}B_{c2}\,d\xi\, u[k] \\ &= Ax[k] + \boldsymbol{B}_1\boldsymbol{w}[k] + B_2u[k] \end{aligned}$$

$$\begin{aligned} z[k](\theta) &= C_{c1}\left(e^{A_c\theta}x[k] + \int_0^\theta e^{A_c(\theta-\xi)}B_{c1}w_c(kh+\xi)\,d\xi + \int_0^\theta e^{A_c\xi}B_{c2}\,d\xi\, u[k]\right) \\ &\quad + D_{c11}w_c(kh+\theta) + D_{c12}u[k] \\ &= C_{c1}e^{A_c\theta}x[k] + C_{c1}\int_0^\theta e^{A_c(\theta-\xi)}B_{c1}w_c(kh+\xi)\,d\xi + D_{c11}w_c(kh+\theta) \\ &\quad + \left(C_{c1}\int_0^\theta e^{A_c\xi}B_{c2}\,d\xi + D_{12}\right)u[k] \\ &= (\boldsymbol{C}_1x[k])(\theta) + (\boldsymbol{D}_{11}\boldsymbol{w}[k])(\theta) + (\boldsymbol{D}_{12}u[k])(\theta) \end{aligned}$$

Lifting Sampled-Data Systems (5/6): Closed-Loop



$$\begin{bmatrix} x_{c\ell}[k+1] \\ \boldsymbol{z}[k] \end{bmatrix} = \begin{bmatrix} A_{c\ell} & \boldsymbol{B}_{c\ell} \\ \boldsymbol{C}_{c\ell} & \boldsymbol{D}_{c\ell} \end{bmatrix}\begin{bmatrix} x_{c\ell}[k] \\ \boldsymbol{w}[k] \end{bmatrix}, \quad x_{c\ell}[k] := \begin{bmatrix} x[k] \\ x_K[k] \end{bmatrix}$$

$$\begin{bmatrix} A_{c\ell} & \boldsymbol{B}_{c\ell} \\ \boldsymbol{C}_{c\ell} & \boldsymbol{D}_{c\ell} \end{bmatrix} := \left[\begin{array}{cc|c} A & 0 & \boldsymbol{B}_1 \\ 0 & 0 & 0 \\ \hline \boldsymbol{C}_1 & 0 & \boldsymbol{D}_{11} \end{array}\right] + \left[\begin{array}{cc} B_2 & 0 \\ 0 & I \\ \hline \boldsymbol{D}_{12} & 0 \end{array}\right]\begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}\left[\begin{array}{cc|c} C_2 & 0 & 0 \\ 0 & I & 0 \end{array}\right]$$

Note: Operators are bounded: $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D} \;\Rightarrow\; \mathbf{L}_2$-stable

Outline

- Motivating Example: Discrete-Time Periodic System
- Lifting Signals
- Lifting Sampled-Data Systems
- Transfer Function of Sampled-Data Systems
  - Frequency Response of Sampled-Data Systems
  - $\mathbf{H}_\infty$-Norm of Sampled-Data Systems

$\mathcal{Z}$-Transformation of Lifted Signals

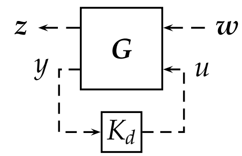
Definition Given lifted signal f

$$\hat{f}[z] = \mathcal{Z}\{f\}[z] := \sum_{k=0}^{\infty} f[k]z^{-k}$$

- Natural extention of $\mathcal{Z}$-trans. for vector-valued signals
- $\hat{f} : \mathbb{C} \to \mathbf{L}_2([0, h], \mathbb{C}^n)$
  cf. $\hat{f} : \mathbb{C} \to \mathbb{C}^n$ for vector-valued signal f

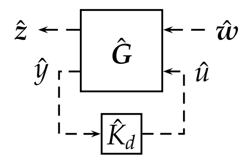
Transfer Function of Sampled-Data Systems

$$\begin{bmatrix} x_{c\ell}[k+1] \\ z[k] \end{bmatrix} = \begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix} \begin{bmatrix} x_{c\ell}[k] \\ w[k] \end{bmatrix}$$



$\mathcal{Z}$-transformation

$$\begin{bmatrix} z\hat{x}_{c\ell}[z] \\ \hat{z}[z] \end{bmatrix} = \begin{bmatrix} A_{c\ell} & B_{c\ell} \\ C_{c\ell} & D_{c\ell} \end{bmatrix} \begin{bmatrix} \hat{x}_{c\ell}[z] \\ \hat{w}[z] \end{bmatrix}$$



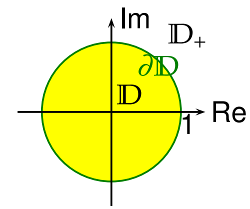
$$\hat{z}[z] = \left(C_{c\ell}(zI - A_{c\ell})^{-1}B_{c\ell} + D_{c\ell}\right)\hat{w}[z] = \mathcal{F}_\ell\left(\hat{G}, \hat{K}_d\right)[z]\hat{w}[z]$$

transfer function

H_∞-Norm of Sampled-Data Systems (1/3)

Notation

- $\mathbb{D} := \{ z \mid z \in \mathbb{C},\ |z| < 1\}$
- $\partial\mathbb{D} := \{ z \mid z \in \mathbb{C},\ |z| = 1\}$
- $\mathbb{D}_+ := \{ z \mid z \in \mathbb{C},\ |z| > 1\}$



- $\mathbb{B} := \left\{ T \;\middle|\; T : \mathbf{L}_2[0,h] \to \mathbf{L}_2[0,h],\ \sup_{u \neq 0} \dfrac{\|Tu\|_2}{\|u\|_2} < \infty \right\}$
- Given $T \in \mathbb{B}$, $\quad \|T\| := \sup_{u\neq 0} \dfrac{\|Tu\|_2}{\|u\|_2}$

H_∞-Norm of Sampled-Data Systems (2/3)

Definition

(i) $\mathbf{H}_\infty(\mathbb{D}_+, \mathbb{B}) := \left\{ \hat{T} \;\middle|\; \hat{T}[z] \in \mathbb{B},\ ^\forall z \in \mathbb{D}_+;\ \sup_{z\in\mathbb{D}_+} \left\|\hat{T}[z]\right\| < \infty \right\}$

(ii) Given $\hat{T} \in \mathbf{H}_\infty(\mathbb{D}_+, \mathbb{B})$,

$$\left\|\hat{T}\right\|_\infty := \sup_{z\in\mathbb{D}_+} \left\|\hat{T}[z]\right\|$$

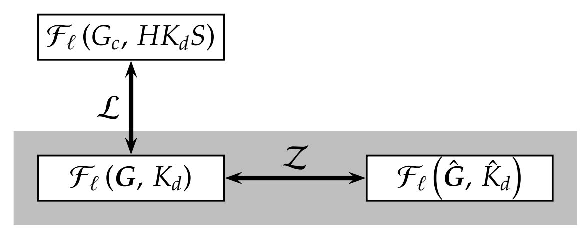
Property $\quad \hat{T}[z] = C_{c\ell}(zI - A_{c\ell})^{-1}B_{c\ell} + D_{c\ell}$

- $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D} \;\Rightarrow\; \hat{T} \in \mathbf{H}_\infty(\mathbb{D}_+, \mathbb{B})$
- Given $\hat{T} \in \mathbf{H}_\infty(\mathbb{D}_+, \mathbb{B})$, $\left\|\hat{T}\right\|_\infty = \sup_{z\in\partial\mathbb{D}} \left\|\hat{T}[z]\right\| = \sup_{\theta\in[0,2\pi)} \left\|\hat{T}[e^{j\theta}]\right\|$

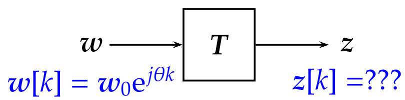
H_∞-Norm of Sampled-Data Systems (3/3)

Theorem $\quad$ Suppose $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D}$

$$\|\mathcal{F}_\ell(G_c, HK_dS)\|_{\mathbf{L}_2\to\mathbf{L}_2} = \|\mathcal{F}_\ell(G, K_d)\|_{\ell_2\to\ell_2} = \left\|\mathcal{F}_\ell\left(\hat{G}, \hat{K}_d\right)\right\|_\infty$$



Frequency Response of Sampled-Data Systems



- Response to sinusoidal input w:

$$w[k] = w_0 \mathrm{e}^{j\theta k}, \quad w_0 \in \mathbf{L}_2[0, h]$$

$$\begin{aligned} z[k] &= C_{c\ell} A_{c\ell}^k x_{c\ell}[0] + C_{c\ell} \sum_{\ell=0}^{k-1} A_{c\ell}^{k-1-\ell} B_{c\ell} w_0 \mathrm{e}^{j\theta\ell} + D_{c\ell} w_0 \mathrm{e}^{j\theta k} \\ &= \hat{T}[\mathrm{e}^{j\theta}] w_0 \mathrm{e}^{j\theta k} + C_{c\ell} A_{c\ell}^k \left(x_{c\ell}[0] - (\mathrm{e}^{j\theta} I - A_{c\ell})^{-1} B_{c\ell} w_0 \right) \end{aligned}$$

- Claim: $z[k] \approx \hat{T}[\mathrm{e}^{j\theta}] w_0 \mathrm{e}^{j\theta k}$ if $\mathrm{eig}(A) \subset \mathbb{D}$ and $k \gg 1$
- Frequency response: $\hat{T}[\mathrm{e}^{j\theta}]$, $\theta \in [0, 2\pi)$