Sampled-Data H_∞ Control

Lecture #6

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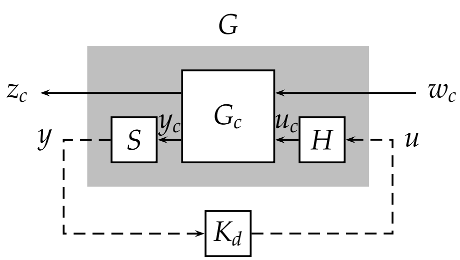
Outline

- Problem Formulation
- Special Case Solution: $G_{c11} = 0$
- General Case Solution
- Computational Aspects

Problem Setup



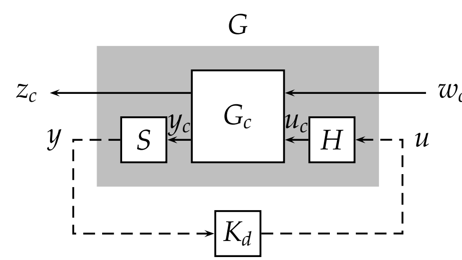
$$G_c : \begin{bmatrix} \dot{x}_c(t) \\ z_c(t) \\ y_c(t) \end{bmatrix} = \begin{bmatrix} A_c & B_{c1} & B_{c2} \\ C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_c(t) \\ w_c(t) \\ u_c(t) \end{bmatrix}$$

$$x_c(t) \in \mathbb{R}^n,\ w_c(t) \in \mathbb{R}^{m_1},\ u_c(t) \in \mathbb{R}^{m_2},\ z_c(t) \in \mathbb{R}^{p_1},\ y_c(t) \in \mathbb{R}^{p_2}$$

$$S : y_c \mapsto y; \quad y[k] = y_c(kh)$$

$$H : u \mapsto u_c; \quad u_c(kh + \theta) = u[k],\ \theta \in [0, h)$$

Problem Formulation



Sampled-Data $\mathbf{H}_\infty$ Control Synthesis	Given G

Find K_d such that

- $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D}$
- $\|\mathcal{F}_\ell(G, K_d)\|_{\mathbf{L}_2 \to \mathbf{L}_2} < 1$

Note that $\|\mathcal{F}_\ell(G, K_d)\|_{\mathbf{L}_2 \to \mathbf{L}_2} = \left\|\mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)\right\|_\infty$

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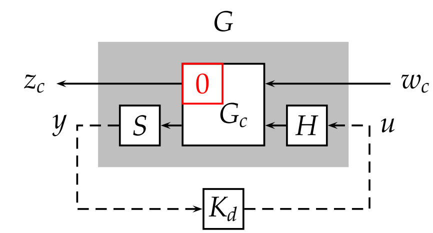
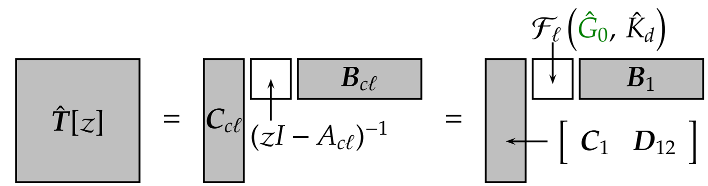
Special Case: $G_{c11} = 0$

Assumption

$G_{c11} = 0$

$\rightarrow$ $\boldsymbol{D}_{11} = 0$



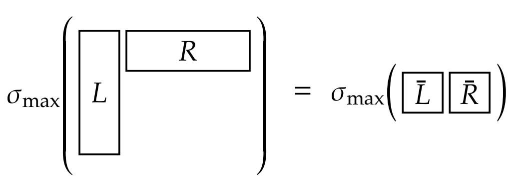


$$\hat{G}_0[z] := \begin{bmatrix} I \\ 0 \\ \hline C_2 \end{bmatrix} (zI - A)^{-1} \begin{bmatrix} I & B_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \\ \hline 0 & 0 \end{bmatrix}$$

Idea for Special Case: $G_{c11} = 0$

Lemma Given $L \in \mathbb{R}^{p \times n}$, $R \in \mathbb{R}^{n \times m}$, $p > n$, $m > n$

$^\exists \bar{L} \in \mathbb{R}^{n \times n}$, $\bar{R} \in \mathbb{R}^{n \times n}$ such that $\sigma_{\max}(LR) = \sigma_{\max}(\bar{L}\bar{R})$



Proof: One such pair is $\bar{L} := \Sigma_L^{\frac{1}{2}} U_L^\mathsf{T}$, $\bar{R} := U_R^\mathsf{T} \Sigma_L^{\frac{1}{2}}$ where

$$L^\mathsf{T} L = U_L \Sigma_L U_L^\mathsf{T}, \quad RR^\mathsf{T} = U_R \Sigma_R U_R^\mathsf{T} \quad \text{(SVD)}$$

Note that $\Sigma_L, U_L, \Sigma_R, U_R \in \mathbb{R}^{n \times n}$

Solution to Special Case: $G_{c11} = 0$ (1/3)

Fact:

- $^\exists B_{s1} \in \mathbb{R}^{n \times m_{s1}}$ $(m_{s1} \le n)$ such that

$$B_{s1} B_{s1}^\mathsf{T} = \boldsymbol{B}_1 \boldsymbol{B}_1^* \in \mathbb{R}^{n \times n}$$

- $^\exists C_{s1} \in \mathbb{R}^{p_{s1} \times n}$, $D_{s12} \in \mathbb{R}^{p_{s1} \times m_2}$ $(p_{s1} \le n + m_2)$ such that

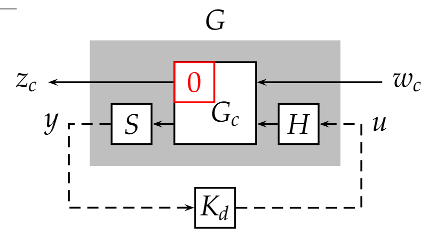
$$\begin{bmatrix} C_{s1}^\mathsf{T} \\ D_{s12}^\mathsf{T} \end{bmatrix} \begin{bmatrix} C_{s1} & D_{s12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_1^* \\ \boldsymbol{D}_{12}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix} \in \mathbb{R}^{(n+m_2) \times (n+m_2)}$$

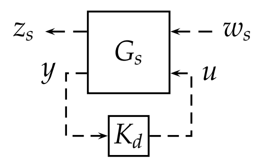
Define:

$$\hat{G}_s[z] := \begin{bmatrix} C_{s1} \\ C_2 \end{bmatrix} (zI - A)^{-1} \begin{bmatrix} B_{s1} & B_2 \end{bmatrix} + \begin{bmatrix} 0 & D_{s12} \\ 0 & 0 \end{bmatrix}$$

Note: Adjoint operator $\langle \boldsymbol{T}x, y \rangle = \langle x, \boldsymbol{T}^* y \rangle$

Solution to Special Case: $G_{c11} = 0$ (2/3)





Lemma Given G with $G_{c11} = 0$, K_d

- $\mathcal{F}_\ell(G, K_d)$ is internally stable
  $\Leftrightarrow$ $\mathcal{F}_\ell(G_s, K_d)$ is internally stable
- Supposing stability

$$\|\mathcal{F}_\ell(G, K_d)\|_{\mathbf{L}_2 \to \mathbf{L}_2} = \left\|\mathcal{F}_\ell\left(\hat{G}_s, \hat{K}_d\right)\right\|_\infty$$

Solution to Special Case: $G_{c11} = 0$ (3/3)

Sketch of Proof: Suppose $\mathrm{eig}(A_{c\ell}) \subset \mathbb{D}$

Fix $\theta \in [0, 2\pi)$, $\boldsymbol{w}_0 \in \mathbf{L}_2[0, h]$ and define $z_0 \in \mathbf{L}_2[0, h]$ by

$$\boldsymbol{z}_0 := \mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)[\mathrm{e}^{j\theta}]\boldsymbol{w}_0$$

Noting that $\mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right) = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix} \mathcal{F}_\ell\left(\hat{G}_0, \hat{K}_d\right)\boldsymbol{B}_1$

$$\begin{aligned} \|z_0\|_2^2 = \boldsymbol{z}_0^*\boldsymbol{z}_0 &= \boldsymbol{w}_0^*\left(\mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)[\mathrm{e}^{j\theta}]\right)^* \begin{bmatrix} \boldsymbol{C}_1^* \\ \boldsymbol{D}_{12}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix} \mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)[\mathrm{e}^{j\theta}]\boldsymbol{w}_0 \\ &= \boldsymbol{w}_0^*\left(\mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)[\mathrm{e}^{j\theta}]\right)^* \begin{bmatrix} C_{s1}^\mathsf{T} \\ D_{s12}^\mathsf{T} \end{bmatrix} \begin{bmatrix} C_{s1} & D_{s12} \end{bmatrix} \mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)[\mathrm{e}^{j\theta}]\boldsymbol{w}_0 \end{aligned}$$

Hence $\left\|\mathcal{F}_\ell\left(\hat{\mathbf{G}}, \hat{K}_d\right)\right\|_\infty = \left\|\begin{bmatrix} C_{s1} & D_{s12} \end{bmatrix} \mathcal{F}_\ell\left(\hat{G}_0, \hat{K}_d\right)\boldsymbol{B}_1\right\|_\infty$

Dual manipulation implies $\left\|\begin{bmatrix} C_{s1} & D_{s12} \end{bmatrix} \mathcal{F}_\ell\left(\hat{G}_0, \hat{K}_d\right)\boldsymbol{B}_1\right\|_\infty = \left\|\mathcal{F}_\ell\left(\hat{G}_s, \hat{K}_d\right)\right\|_\infty$

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Idea for General Case: Loop Shifting (1/2)

Lemma $\hat{G}[z] = C(zI - A)^{-1}B + D$. SAE

- $\mathrm{eig}(A) \subset \mathbb{D}$ and $\left\|\hat{G}\right\|_\infty < 1$
- $\sigma_{\max}(D) < 1$, $\mathrm{eig}(\bar{A}) \subset \mathbb{D}$ and $\left\|\hat{G}_{\mathrm{LS}}\right\|_\infty < 1$

$$\hat{G}_{\mathrm{LS}}[z] := \bar{C}(zI - \bar{A})^{-1}\bar{B}, \quad \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & 0 \end{bmatrix} := \begin{bmatrix} A + BR^{-1}D^\mathsf{T}C & BR^{-\frac{1}{2}} \\ (I - DD^\mathsf{T})^{-\frac{1}{2}}C & 0 \end{bmatrix}$$

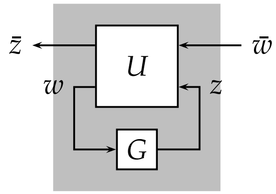
where $R := I - D^\mathsf{T}D > 0$

Remark: $\left\|\hat{G}\right\|_\infty \neq \left\|\hat{G}_{\mathrm{LS}}\right\|_\infty$

Idea for General Case: Loop Shifting (2/2)

Sketch of Proof: Let U: unitary

$$\begin{bmatrix} \bar{z} \\ w \end{bmatrix} = U \begin{bmatrix} \bar{w} \\ z \end{bmatrix} \quad \Rightarrow \quad \|\bar{z}\|_2^2 - \|\bar{w}\|_2^2 = \|z\|_2^2 - \|w\|_2^2$$



$$\|\hat{G}\|_\infty < 1 \Leftrightarrow \|\bar{z}\|_2^2 - \|\bar{w}\|_2^2 < 0 \Leftrightarrow \|z\|_2^2 - \|w\|_2^2 < 0 \Leftrightarrow \left\|\mathcal{F}_\ell\left(U, \hat{G}\right)\right\|_\infty < 1$$

G_{LS} is recovered by $U_D = \begin{bmatrix} -D & (I - DD^\mathsf{T})^{\frac{1}{2}} \\ (I - D^\mathsf{T}D)^{\frac{1}{2}} & D^\mathsf{T} \end{bmatrix}$

Reduction to Discrete-Time Problem (1/3)

Suppose $\|\boldsymbol{D}_{11}\| < 1$ $\leftarrow$ otherwise no $\mathbf{H}_\infty$ control exists

Fact: $(I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{-1}$, $(I - \boldsymbol{D}_{11}\boldsymbol{D}_{11}^*)^{-1}$ are weill-defined and PD

- $\exists B_{s1} \in \mathbb{R}^{n \times m_{s1}}$ $(m_{s1} \le n)$ such that $B_{s1}B_{s1}^\mathsf{T} = \boldsymbol{B}_1(I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{-1}\boldsymbol{B}_1^* \in \mathbb{R}^{n \times n}$
- $\exists C_{s1} \in \mathbb{R}^{p_{s1} \times n}$, $D_{s12} \in \mathbb{R}^{p_{s1} \times m_2}$ $(p_{s1} \le n + m_2)$ such that

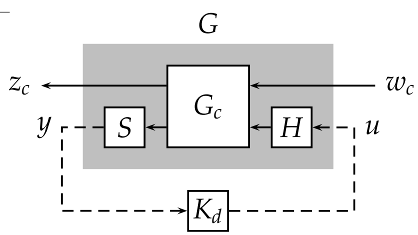
$$\begin{bmatrix} C_{s1}^\mathsf{T} \\ D_{s12}^\mathsf{T} \end{bmatrix} \begin{bmatrix} C_{s1} & D_{s12} \end{bmatrix} = \begin{bmatrix} \boldsymbol{C}_1^* \\ \boldsymbol{D}_{12}^* \end{bmatrix} (I - \boldsymbol{D}_{11}\boldsymbol{D}_{11}^*)^{-1} \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix} \in \mathbb{R}^{(n+m_2) \times (n+m_2)}$$

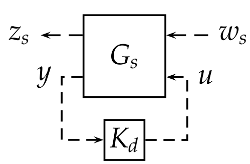
Define:

$$\hat{G}_s[z] := \begin{bmatrix} C_{s1} \\ C_2 \end{bmatrix} (zI - A_s)^{-1} \begin{bmatrix} B_{s1} & B_{s2} \end{bmatrix} + \begin{bmatrix} 0 & D_{s12} \\ 0 & 0 \end{bmatrix}$$

where $\begin{bmatrix} A_s & B_s \end{bmatrix} := \begin{bmatrix} A & B_2 \end{bmatrix} + \boldsymbol{B}_1(I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{-1}\boldsymbol{D}_{11}^* \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$

Reduction to Discrete-Time Problem (2/3)





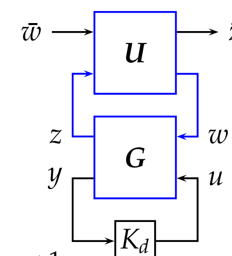
Theorem Given G with $\|\boldsymbol{D}_{11}\| < 1$, K_d. SAE

- $\mathcal{F}_\ell(G, K_d)$ is internally stable and $\|\mathcal{F}_\ell(G, K_d)\|_{\mathbf{L}_2 \to \mathbf{L}_2} < 1$
- $\mathcal{F}_\ell(G_s, K_d)$ is internally stable and $\left\|\mathcal{F}_\ell\left(\hat{G}_s, \hat{K}_d\right)\right\|_\infty < 1$

Message: G and G_s are "equivalent" in $\mathbf{H}_\infty$ control sense

Reduction to Discrete-Time Problem (3/3)

Sketch of Proof:



Step 1: Eliminate $\boldsymbol{D}_{11}$

Define $\boldsymbol{U}$: unitary

$$\boldsymbol{U} := \begin{bmatrix} -\boldsymbol{D}_{11} & (I - \boldsymbol{D}_{11}\boldsymbol{D}_{11}^*)^{\frac{1}{2}} \\ (I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{\frac{1}{2}} & \boldsymbol{D}_{11}^* \end{bmatrix}$$

$$\left\|\mathcal{F}_\ell\left(\hat{\boldsymbol{G}}, \hat{K}_d\right)\right\|_\infty < 1 \quad \Leftrightarrow \quad \left\|\mathcal{F}_\ell\left(\boldsymbol{U}, \mathcal{F}_\ell\left(\hat{\boldsymbol{G}}, \hat{K}_d\right)\right)\right\|_\infty < 1$$

Notice $\mathcal{F}_\ell\left(\boldsymbol{U}, \mathcal{F}_\ell\left(\hat{\boldsymbol{G}}, \hat{K}_d\right)\right) = \mathcal{F}_\ell\left(\boldsymbol{U} \star \hat{\boldsymbol{G}}, \hat{K}_d\right)$

$$(\boldsymbol{U} \star \hat{\boldsymbol{G}})[z] = \begin{bmatrix} (I - \boldsymbol{D}_{11}\boldsymbol{D}_{11}^*)^{-\frac{1}{2}}\boldsymbol{C}_1 \\ C_2 \end{bmatrix} (zI - A_s)^{-1} \begin{bmatrix} \boldsymbol{B}_1(I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{-\frac{1}{2}} & B_{s2} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & (I - \boldsymbol{D}_{11}\boldsymbol{D}_{11}^*)^{-\frac{1}{2}}\boldsymbol{D}_{12} \\ 0 & 0 \end{bmatrix}$$

Step 2: Invoke Lemma for special case

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Rest of Our Task

What we know: $\exists$ DT system equivalent to SD one

What we don't know: How to

- check if $\|\boldsymbol{D}_{11}\| < 1$
- compute
  - $\boldsymbol{B}_1(I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{-1}\boldsymbol{B}_1^*$
  - $\begin{bmatrix} \boldsymbol{C}_1^* \\ \boldsymbol{D}_{12}^* \end{bmatrix}(I - \boldsymbol{D}_{11}\boldsymbol{D}_{11}^*)^{-1}\begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$
  - $\boldsymbol{B}_1(I - \boldsymbol{D}_{11}^*\boldsymbol{D}_{11})^{-1}\boldsymbol{D}_{11}^*\begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix}$

  or $\begin{bmatrix} \boldsymbol{B}_1 & 0 \\ 0 & \begin{bmatrix} \boldsymbol{C}_1^* \\ \boldsymbol{D}_{12}^* \end{bmatrix} \end{bmatrix}\begin{bmatrix} I & -\boldsymbol{D}_{11}^* \\ -\boldsymbol{D}_{11} & I \end{bmatrix}^{-1}\begin{bmatrix} \boldsymbol{B}_1^* & 0 \\ 0 & \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{D}_{12} \end{bmatrix} \end{bmatrix}$

Systems with Two Point Boundary Conditions

SSBC (State-Space Systems with Two-Point Boundary Conditions)

$$\Sigma : \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \Omega x(0) + \Upsilon x(h) = 0$$

- well-posed ($u = 0 \Rightarrow x = 0$) $\Leftrightarrow$ $\Xi := \Omega + \Upsilon e^{Ah}$: nonsingular
- $\mathbf{L}_2[0, h] \to \mathbf{L}_2[0, h]$

Impulse Operator $\mathcal{I}_\theta$

- $\mathbb{R}^n \ni v \mapsto \mathcal{I}_\theta v := \delta(t - \theta)v$
- $\Sigma\mathcal{I}_\theta$: $\mathbb{R}^m \to \mathbf{L}_2[0, h]$ if $D = 0$

Sample Operator $\mathcal{S}_\theta$

- $C[0, h] \ni f \mapsto \mathcal{S}_\theta f := f(\theta)$
- $\mathcal{S}_\theta\Sigma$: $\mathbf{L}_2[0, h] \to \mathbb{R}^p$ if $D = 0$

SSBC Representation

Fact: SSBC is a generalization of operators in lifted systems

In fact

$$\begin{bmatrix} A & \boldsymbol{B}_1 & B_2 \\ \boldsymbol{C}_1 & \boldsymbol{D}_{11} & \boldsymbol{D}_{12} \end{bmatrix} = \begin{bmatrix} \mathcal{S}_h & 0 \\ 0 & I \end{bmatrix}\Sigma_{G_c}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \Sigma_H\mathcal{I}_0 \end{bmatrix}$$

where

$$\Sigma_{G_c} : \begin{bmatrix} \dot{x}_c(t) \\ x_c(t) \\ z_c(t) \end{bmatrix} = \begin{bmatrix} A_c & I & B_{c1} & B_{c2} \\ I & 0 & 0 & 0 \\ C_{c1} & 0 & D_{c11} & D_{c12} \end{bmatrix}\begin{bmatrix} x_c(t) \\ \xi(t) \\ w_c(t) \\ u_c(t) \end{bmatrix}, \quad x_c(0) = 0$$

$$\Sigma_H : \begin{bmatrix} \dot{x}_H(t) \\ u_c(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} x_H(t) \\ v(t) \end{bmatrix}, \quad x_H(0) = 0$$

Formulas for SSBCs (1/3)

- **Adjoint system**:

$$\Sigma^* \quad : \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -A^\mathsf{T} & C^\mathsf{T} \\ -B^\mathsf{T} & D^\mathsf{T} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

$$e^{A^\mathsf{T} h} \Upsilon^\mathsf{T} (\Xi^\mathsf{T})^{-1} x(0) + \Omega^\mathsf{T} (\Xi^\mathsf{T})^{-1} e^{A^\mathsf{T} h} x(h) = 0$$

- **Inversion**: Exists iff $\det(D) \neq 0$ and $\det(\Omega + \Upsilon e^{(A - BD^{-1}C)h}) \neq 0$

$$\Sigma^{-1} \quad : \quad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix},$$

$$\Omega x(0) + \Upsilon x(h) = 0$$

Formulas for SSBCs (2/3)

- **Addition**:

$$\Sigma_1 + \Sigma_2 \quad : \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \hline y(t) \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1 + D_2 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \hline u(t) \end{bmatrix},$$

$$\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix} \begin{bmatrix} x_1(h) \\ x_2(h) \end{bmatrix} = 0$$

- **Multiplication**:

$$\Sigma_1 \Sigma_2 \quad : \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \hline y(t) \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ \hline u(t) \end{bmatrix},$$

$$\begin{bmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \Upsilon_1 & 0 \\ 0 & \Upsilon_2 \end{bmatrix} \begin{bmatrix} x_1(h) \\ x_2(h) \end{bmatrix} = 0$$

Formulas for SSBCs (3/3)

- For Σ with $D = 0$
  - $(\Sigma \mathcal{I}_\theta)^* = \mathcal{S}_\theta \Sigma^*$
  - $(\mathcal{S}_\theta \Sigma)^* = \Sigma^* \mathcal{I}_\theta$
  - $\mathcal{S}_h \Sigma \mathcal{I}_0 = C e^{Ah} \Xi^{-1} \Omega B$
  - $\mathcal{S}_0 \Sigma \mathcal{I}_h = -C \Xi^{-1} \Upsilon B$
- For Σ with $D = 0$ and $CB = 0$
  - $\mathcal{S}_0 \Sigma \mathcal{I}_0 = C \Xi^{-1} \Omega B$
  - $\mathcal{S}_h \Sigma \mathcal{I}_h = -C e^{Ah} \Xi^{-1} \Upsilon B$

Proof: Use the fact

$$y(t) = \int_0^h K(t, s) u(s)\, ds + Du(t), \quad K(t, s) := \begin{cases} C e^{At} \Xi^{-1} \Omega e^{-As} B, & 0 \le s < t \le h \\ -C e^{At} \Xi^{-1} \Upsilon e^{A(h-s)} B, & 0 \le t < s \le h \end{cases}$$

Computational Formula for G_s

Theorem Given G with $\|\boldsymbol{D}_{11}\| < 1$, $D_{c11} = 0$

$$\begin{bmatrix} B_{s1} B_{s1}^\mathsf{T} & \begin{bmatrix} A_s & B_{s2} \end{bmatrix} \\ \begin{bmatrix} A_s^\mathsf{T} \\ B_{s2}^\mathsf{T} \end{bmatrix} & \begin{bmatrix} C_{s1}^\mathsf{T} \\ D_{s12}^\mathsf{T} \end{bmatrix} \begin{bmatrix} C_{s1}^\mathsf{T} \\ D_{s12}^\mathsf{T} \end{bmatrix}^\mathsf{T} \end{bmatrix} = \left[\begin{array}{ccc} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \hline 0 & 0 & -I \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} \end{array}\right] \left[\begin{array}{c|cc} 0 & I & 0 \\ 0 & 0 & I \\ -\Gamma_{33}^{-1} & -\Gamma_{33}^{-1}\Gamma_{31} & -\Gamma_{33}^{-1}\Gamma_{32} \end{array}\right]$$

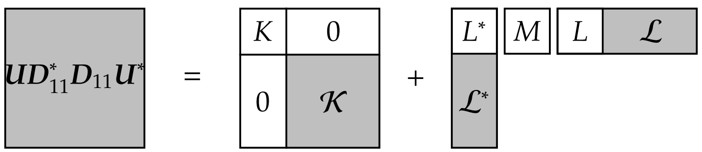
where

$$\left[\begin{array}{cc|cc} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 \\ 0 & I & 0 & 0 \\ \hline \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & I \end{array}\right] := \exp\left(\begin{bmatrix} A_u & -B_u B_u^\mathsf{T} \\ C_u^\mathsf{T} C_u & -A_u^\mathsf{T} \end{bmatrix} h \right)$$

$$\begin{bmatrix} A_u & B_u \\ C_u & * \end{bmatrix} := \left[\begin{array}{cc|c} A_c & B_{c2} & B_{c1} \\ 0 & 0 & 0 \\ \hline C_{c1} & D_{c12} & * \end{array}\right]$$

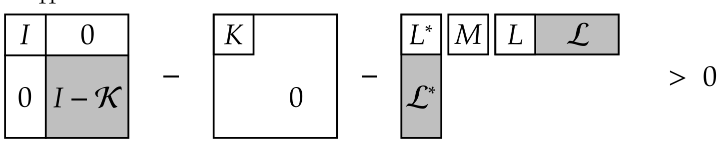
Idea to Check $\|D_{11}\| < 1$ (1/3)

Suppose $^\exists \boldsymbol{U}$: unitary such that

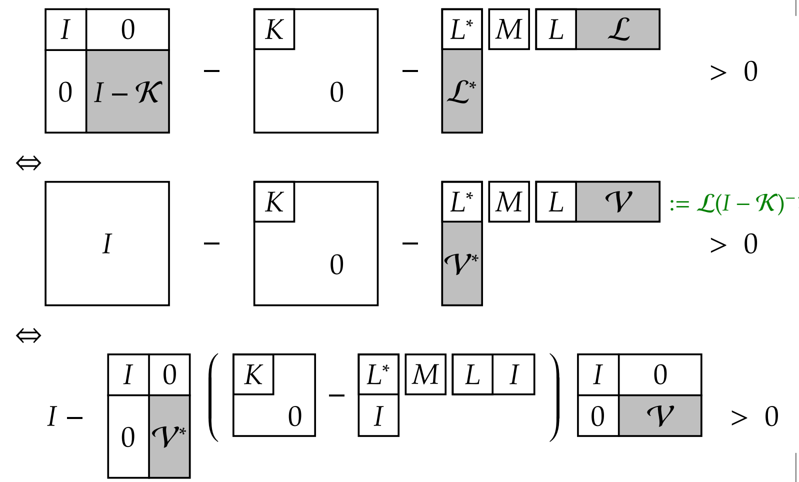


and $I - \mathcal{K} > 0$

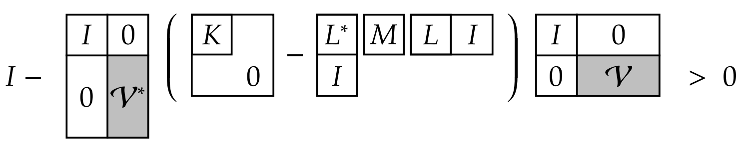
$I - D_{11}^* D_{11} > 0 \quad \Leftrightarrow$



Idea to Check $\|D_{11}\| < 1$ (2/3)



Idea to Check $\|D_{11}\| < 1$ (3/3)



$\Leftrightarrow$

$$\rho\left(\begin{bmatrix} I & 0 \\ 0 & W \end{bmatrix}\left(\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L^* \\ I \end{bmatrix} M \begin{bmatrix} L & I \end{bmatrix}\right)\right) < 1$$

$$W := \mathcal{V}\mathcal{V}^* = \mathcal{L}(I-\mathcal{K})^{-1}\mathcal{L}^*$$

- $^\exists \boldsymbol{U}$?
- How to compute W, K, M, L ?

Preliminaries for Algorithm (1/3)

Suppose $\sigma_{\max}(D_{c11}) < 1 \quad \leftarrow$ otherwise $\|D_{11}\| \geq 1$

Given $\theta \in [0, 2\pi)$. Let $\Psi : \mathbf{L}_2[0,h] \to \ell_2$,

$$(\Psi f)[k] := \frac{1}{\sqrt{h}} \int_0^h e^{-j\omega_k t} f(t)\, dt,$$

$$\omega_k := \frac{2\pi v_k + \theta}{h}, \quad v_k := \{0, 1, -1, 2, -2, \ldots\}$$

Claim: Ψ is unitary

Preliminaries for Algorithm (2/3)

Lemma Suppose $e^{j\theta} \notin \mathrm{eig}(A)$. $y = \Psi \boldsymbol{D}_{11}^* \boldsymbol{D}_{11} \Psi^* u$

$$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \end{bmatrix} = \left(\begin{bmatrix} K_0 & & & 0 \\ & K_1 & & \\ & & K_2 & \\ 0 & & & \ddots \end{bmatrix} + \begin{bmatrix} L_0^* \\ L_1^* \\ L_2^* \\ \vdots \end{bmatrix} M \begin{bmatrix} L_0 & L_1 & L_2 & \cdots \end{bmatrix} \right) \begin{bmatrix} u[0] \\ u[1] \\ u[2] \\ \vdots \end{bmatrix}$$

$$K_k := \hat{G}_{c11}^*(j\omega_k)\hat{G}_{c11}(j\omega_k), \quad L_k := \begin{bmatrix} -(j\omega_k I - A_c)^{-1} B_{c1} \\ (j\omega_k I - A_c)^{-*} C_{c1}^{\mathsf{T}} \hat{G}_{c11}(j\omega_k) \end{bmatrix}$$

$$M := \frac{1}{h} \begin{bmatrix} Q & e^{j\theta}(e^{j\theta} I - A)^* \\ e^{-j\theta}(e^{j\theta} I - A) & 0 \end{bmatrix}, \quad Q := \int_0^h e^{A_c^{\mathsf{T}} t} C_{c1}^{\mathsf{T}} C_{c1} e^{A_c t}\, dt$$

Proof: (complicated but) straightforward

Preliminaries for Algorithm (3/3)

$$A_H := \begin{bmatrix} -A_c^{\mathsf{T}} & -C_{c1}^{\mathsf{T}} C_{c1} \\ 0 & A_c \end{bmatrix} + \begin{bmatrix} -C_{c1}^{\mathsf{T}} D_{c11} \\ B_{c1} \end{bmatrix} (I - D_{c11}^{\mathsf{T}} D_{c11})^{-1} \begin{bmatrix} B_{c1}^{\mathsf{T}} & D_{c11}^{\mathsf{T}} C_{c1} \end{bmatrix}$$

Lemma Let $\Omega := \{\omega : \ \omega \in \mathbb{R},\ j\omega \in \mathrm{eig}(A_H)\}$

If $\Omega \neq \emptyset$ then $I - K_k > 0$, $^\forall k \geq N+1$ where $N \in \mathbb{Z}_+$ satisfies

$$|\omega_{N+1}| > \max\{|\omega| : \ \omega \in \Omega\}, \quad |\omega_{N+2}| > \max\{|\omega| : \ \omega \in \Omega\}$$

else ($\Omega = \emptyset$), $I - K_k > 0$, $^\forall k \in \mathbb{Z}_+$.

Proof: $\sigma_{\max}(D_{c11}) < 1 \ \Rightarrow \ I - K_k > 0$ for $k \gg 1$

$\sigma_{\max}(G_{c11}(j\omega)) = 1$ for $\omega \in \Omega$

Algorithm to Check $\|D_{11}\| < 1$ (1/2)

Algorithm **Step 0**: If $\sigma_{\max}(D_{c11}) \geq 1$ then $\|\boldsymbol{D}_{11}\| \geq 1$. Stop

Step 1: Fix $\theta \in [0, 2\pi)$ so that $e^{j\theta} \notin \mathrm{eig}(A)$, $e^{j\theta} \notin \mathrm{eig}(e^{A_H h})$

$$W_\infty := \frac{1}{2} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} (e^{j\theta} I - e^{A_H h})^{-1} (e^{j\theta} I + e^{A_H h})$$
$$- \frac{1}{2} \begin{bmatrix} I & 0 \\ 0 & (e^{j\theta} I - A)^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & -(e^{j\theta} I + A) \\ -(e^{j\theta} I + A)^* & 2Q \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (e^{j\theta} I - A)^{-1} \end{bmatrix}$$

Step 2: If $\Omega \neq \emptyset$, fix N as above.

$$K := \begin{bmatrix} K_0 & & 0 \\ & \ddots & \\ 0 & & K_N \end{bmatrix}, \quad L := \begin{bmatrix} L_0 & \cdots & L_N \end{bmatrix}$$

Algorithm to Check $\|D_{11}\| < 1$ (2/2)

$$W_N := -\sum_{k=0}^{N} \left(\begin{bmatrix} C_{c1}^{\mathsf{T}} C_{c1} & (j\omega_k I - A_c)^* \\ j\omega_k I - A & 0 \end{bmatrix} + \begin{bmatrix} -C_{c1}^{\mathsf{T}} D_{c11} \\ B_{c1} \end{bmatrix} (I - D_{c11}^{\mathsf{T}} D_{c11})^{-1} \begin{bmatrix} -C_{c1}^{\mathsf{T}} D_{c11} \\ B_{c1} \end{bmatrix}^{\mathsf{T}} \right)^{-1}$$
$$+ \sum_{k=0}^{N} \begin{bmatrix} I & 0 \\ 0 & (j\omega_k I - A_c)^{-1} \end{bmatrix}^* \begin{bmatrix} 0 & I \\ I & -C_{c1}^{\mathsf{T}} C_{c1} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & (j\omega_k I - A_c)^{-1} \end{bmatrix}$$

$$\Gamma := \left(\begin{bmatrix} K & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} L^* \\ I \end{bmatrix} M \begin{bmatrix} L & I \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & W_\infty - W_N \end{bmatrix}$$

Otherwise ($\Omega = \emptyset$), $\Gamma := M W_\infty$

Step 3: If $\rho(\Gamma) < 1$ then $\|\boldsymbol{D}_{11}\| < 1$ else $\|\boldsymbol{D}_{11}\| \geq 1$

Proof: Straightforward. Note that

$$\sum_{i=0}^{\infty} (j\omega_i - A_c)^{-1} = \frac{1}{2}(e^{j\theta} - A)^{-1}(e^{j\theta} I + A)$$

Recap

Procedure for Sampled-Data $\mathbf{H}_\infty$ Control Synthesis:

- Step 1: Check if $\|D_{11}\| < 1$
- Step 2: Compute G_s
- Step 3: Discrete-time $\mathbf{H}_\infty$ control synthesis for G_s

References

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- Mirkin, Palmor, IEEE Trans. Automatic Control, vol. 44, pp. 833–840 (1999)
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