Research Statement

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1 Overview

My work is in:
• Analytic number theory and its applications in mathematical physics, especially quantum chaos.
• Nodal lines and zeros of random functions.
• Probability, stochastic processes and random fields.

2 Detailed description

2.1 Lattice counting problems

During my PhD studies, I worked on a problem in analytic number theory related to a recently developed field, usually referred as quantum chaos. Quantum chaos is an interdisciplinary branch of physics which is a meeting point of mathematical physics, number theory and other disciplines.

One of the outstanding problems in this subject concerning the statistics of energy levels of quantum systems in which one studies the counting function for the number of levels in a random energy window. In the case of the free motion of a particle on the torus, this equals the number of integer points in an annulus. The latter is a classical problem in number theory, called the Gauss circle problem.

In some regimes, one finds a nonuniversal distribution. For some other regimes, it has been conjectured by Bleher and Lebowitz that its distribution is Gaussian. In this research, I proved the conjecture for irrational tori with some additional "generic" properties. The technique I apply is a mixture of number theory and analysis.

One of the obstacles in my approach is the existence of so-called "close pairs" of lattice points, which correspond to small values of a quadratic form. To bound the rate of occurrence of this phenomenon, I use a recent result due to Eskin-Margulis-Mozes, where they prove a quantitative version of the Oppenheim conjecture, an important conjecture in number theory on the distribution of values of quadratic forms. To suit my situation I proved a uniform version of their result.

In the future I wish to continue working in this field. Despite our efforts, the conjecture of Bleher and Lebowitz is still open. It would be very challenging to prove this conjecture in full generality or at least find some other examples of systems where it holds.

2.2 Nodal lines on manifolds and generic billiards

Nodal patterns (first described by Ernest Chladni in 18th century) appear in many problems in engineering, physics and natural sciences: they describe the sets that remain stationary during vibrations, hence their importance in such diverse areas as musical instruments, mechanical structures, earthquake study and other areas. They also arise in the study of wave propagation, and in astrophysics; this is a very active and rapidly developing research area. Let $\mathcal{M}$ be a compact surface (for example S^2, the two dimensional unit sphere), and $\phi : \mathcal{M} \rightarrow \mathbb{R}$ be a real valued function. The nodal line of ϕ is its zero set $\{x \in \mathcal{M} : \phi(x) = 0\}$.

Many fundamental PDEs in mathematics and physics (such as heat and wave equations) are studied using separation of variables, that involves expansions in series of eigenfunctions of Laplace-type operators (e.g. Fourier series), thus the study of spectra and eigenfunctions of the Laplacian are of fundamental importance in mathematics and physics. It is known that the spectrum $\{\lambda_j\}_{0\le j<\infty}$ of the negative Laplacian $-\Delta$ is a discrete infinite subset of $\mathbb{R}$, so that $\lambda_j \rightarrow \infty$. Let ϕ_j be the eigenfunction corresponding to λ_j.

It is particularly interesting to study the behavior of high energy eigenfunctions (that correspond to highly excited membranes), including their nodal lines. A basic question concerns the length l_j of the nodal line of ϕ_j. S.T. Yau conjectured that that the length is commensurable with the square root of the eigenvalue i.e. that there exist constants $c_{\mathcal{M}}, C_{\mathcal{M}} > 0$ so that

$$c_{\mathcal{M}}\sqrt{\lambda_j} < l_j < C_{\mathcal{M}}\sqrt{\lambda_j}.$$

Yau's conjecture was resolved by Donnely and Fefferman for real-analytic surfaces, but is open in general.

Together with Zeév Rudnick, I studied the length of the nodal lines on the torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ in more detail. Here, the multiplicities of the eigenvalues allow us to introduce a notion of *Gaussian random eigenfunction* and study the distribution of the length of their nodal sets. Our results imply, in particular, that for a "typical" eigenfunction ϕ_j,

$$\mathit{length}\big(\{\phi_j = 0\}\big) \sim c\sqrt{\lambda_j}$$

for some explicitly given c. Our technique is a mixture of probability and the theory of stochastic processes, and number theory. One of the key ingredients is the arithmetic of quadratic forms.

The sphere $\mathcal{S}^2 \subset \mathbb{R}^3$ is another example of a surface with highly degenerate Laplace spectrum allowing us to endow the eigenspaces with Gaussian measure. The eigenvalues behave much more regularly than on arithmetic tori, but the curved metric on the sphere presents additional difficulties. Recently, I studied the nodal length of random eigenfunctions on the sphere. In a work in progress I was able to establish precise asymptotic expression for the variance of the nodal length, a very significant advance in this area.

The *logarithmic* asymptotics for the variance I found was a major surprise, contradicting the *linear* prediction by some experts in this field. This behaviour is however consistent with Michael Berry's predictions for the same quantity in *chaotic* dynamical systems (which do not include the sphere). For the spherical case, as well as in Berry's original research, one observes a new phenomenon we refer to as "Berry's cancelation phenomenon", which is responsible for the variance being smaller than what a natural conjecture would be.

Back to the torus, in a work in progress with Manjunath Krishnapur, we were able to observe "Berry's cancelation phenomenon" as well, and formulate a conjecture for precise asymptotic form of the variance for "generic" sequence of energy levels (which is different than in the spherical case); it is plausible that we will be able to prove it in the near future. Our results show that "Berry's cancelation phenomenon" is of more general nature, which we would like to study and understand more in the future. Following this recent progress, it seems possible to study the nodal length on "generic" surfaces. In a work in progress I study that for "generic" surfaces, together with John Toth. It seems plausible that the variance is "generically" logarithmic.

In the case of a billiard, one is interested in the number of intersections of the nodal lines with the boundary. In applications, one is often interested not only in eigenfunctions ("pure states"), but also in their linear combinations ("wavepackets"). In a joint research with John Toth, I considered random combinations

$$\phi(x) = \sum_{n=1}^{N} a_n \phi_j(x)$$

of eigenfunctions, where a_n are standard Gaussian i.i.d. We were able to compute precise asymptotics for the expected number of intersections of the nodal line with the boundary of the billiard.

In the future I would like to study the distribution of random nodal lines in more detail.

2.3 Zeros of random trigonometric polynomials

As a model for the higher dimensional case, together with Andrew Granville, I considered the 1-dimensional case. Here we are interested in the number of zeros of a *"random function"*, where the latter notion needs to be defined precisely.

The most natural example of ensemble of random functions, originally studied by Littlewood and Offord, and Kac, is of algebraic polynomials

$$P_N(x) = a_0 + a_1 x + \dots a_N x^N$$

of large degree N, with Gaussian i.i.d coefficients a_n. They proved that with "high" probability, the distribution of the number of roots of P_N is concentrated around $const \cdot \log N$. Maslova proved that the distribution of the number of zeros of polynomials tends to Gaussian, with expectation and variance proportional to $\log N$, consistent with the earlier observation.

In our research, we considered the *trigonometric* polynomials

$$T_N(t) = \sum_{n=1}^{N} \big(a_n \cos(nt) + b_n \sin(nt)\big)$$

with a_n and b_n standard Gaussian i.i.d. The distribution of zeros of the trigonometric polynomials occurs in a variety of problems in science and engineering, such as nuclear physics (in particular, random matrix theory), statistical mechanics, quantum mechanics, theory of noise etc.

It is known that the expected value of the number of zeros of T_N is asymptotic to $\frac{2}{\sqrt{3}}N$. Qualls and later Farahmand gave various upper bounds for the variance. Bogomolny, Bohigas and Leboeuf predicted that the variance is asymptotic to cN for some constant $c \approx 0.558\ldots$. We confirmed their prediction, and moreover established a central limit theorem for the number of zeros of T_N. To obtain our results we apply techniques from probability theory and stochastic processes, and harmonic analysis.

In the future I wish to continue studying the distribution of zeros of random functions for other ensembles. Our recent success gives rise to a hope that we may be able to generalize the results obtained.