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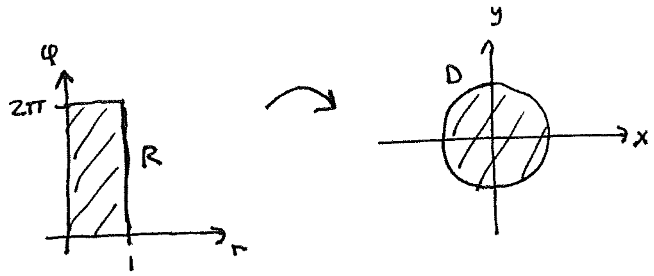
$\boxed{6.20}$ LÅT $D = \{(x,y) : x^2+y^2<1\}$, BERÄKNA

$$I := \iint_D \frac{(x+y)^2}{1+x^2+y^2}\,dx\,dy = \iint_D f(x,y)\,dx\,dy$$

BYT TILL POLÄRA KOORDINATER

$$\begin{cases} x = r\cdot\cos\varphi =: g(r,\varphi) \\ y = r\cdot\sin\varphi =: h(r,\varphi) \end{cases}$$



ENLIGT SATS 6 (s. 261)

$$I = \iint_D f(x,y)\,dx\,dy = \iint_R f(g(r,\varphi), h(r,\varphi))\cdot|J(r,\varphi)|\cdot dr\,d\varphi$$

DÄR

$$|J(r,\varphi)| = \left|\frac{d(x,y)}{d(r,\varphi)}\right| = \begin{vmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \varphi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$= |r\cdot\cos^2\varphi + r\sin^2\varphi| = |r(\cos^2\varphi+\sin^2\varphi)| = r$$

SÅLEDES

$$I = \int_{r=0}^{1}\int_{\varphi=0}^{2\pi} \frac{\overbrace{(r\cdot\cos\varphi + r\sin\varphi)^2}^{=r^2(\cos^2\varphi+\sin^2\varphi+2\cos\varphi\sin\varphi)}}{1+r^2}\cdot r\,dr\,d\varphi$$

$$= \int_{r=0}^{1}\int_{\varphi=0}^{2\pi} \frac{r^3}{1+r^2}\cdot(1+\underbrace{2\cos\varphi\sin\varphi}_{=\sin(2\varphi)})\,dr\,d\varphi$$

POLYNOMDIVISION:

$$\begin{array}{r|l} & r \\ \hline 1+r^2 & r^3 \\ & -(r+r^3) \\ \hline & -r \end{array} \Rightarrow \frac{r^3}{1+r^2} = r - \frac{r}{1+r^2}$$

$$= \int_{r=0}^{1} \frac{r^3}{1+r^2}\,dr\cdot \underbrace{\int_{\varphi=0}^{2\pi}(1+\sin 2\varphi)\,d\varphi}_{=\int_0^{2\pi} d\varphi + \underbrace{\int_0^{2\pi}\sin 2\varphi\,d\varphi}_{=0} = 2\pi}$$

$$= \int_0^1\left(r - \frac{r}{1+r^2}\right)dr\cdot 2\pi = 2\pi\cdot\left[\frac{r^2}{2} - \frac{1}{2}\ln(1+r^2)\right]_0^1 = 2\pi\cdot\left(\frac{1}{2}-\frac{1}{2}\ln 2\right)$$

$$= \pi\cdot(1-\ln 2)$$

$\boxed{6.19}$ LÅT $D = \{ (x,y) : |x| + |y| \le 1 \}$, BERÄKNA

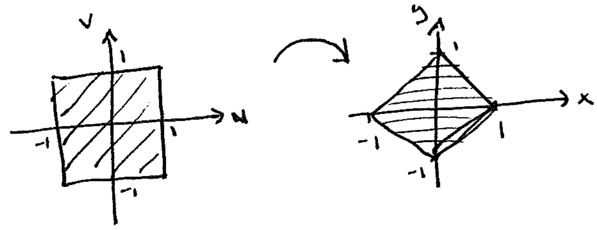
$$I := \iint_D (x^2 - y^2)^{10} \, dx\,dy$$

BYT KOORDINATER

$$\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases} \iff \begin{cases} x+y = u \\ x-y = v \end{cases} \qquad \left| \frac{d(x,y)}{d(u,v)} \right| = \left| \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} \right| = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

BESTÄM INTEGRATIONS-OMRÅDE GENOM ATT BETRAKTA RANDEN AV D:

$$\begin{aligned} x + y &= 1 \implies u = 1 \\ -x - y &= 1 \implies u = -1 \\ x - y &= 1 \implies v = 1 \\ -x + y &= 1 \implies v = -1 \end{aligned}$$



VI FÅR

$$I = \int_{u=-1}^{1} \int_{v=-1}^{1} \left(\frac{1}{4}(u+v)^2 - \frac{1}{4}(u-v)^2 \right)^{10} \cdot \frac{1}{2} \cdot du\,dv = \frac{1}{2} \int_{u=-1}^{1} \int_{v=-1}^{1} (u \cdot v)^{10} du\,dv$$

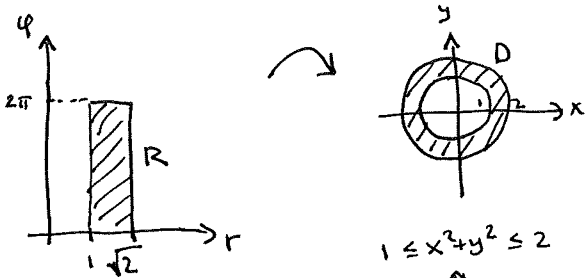
$$= \frac{1}{2} \left(\left[\frac{u^{11}}{11} \right]_{-1}^{1} \right)^2 = \frac{1}{2} \left(\frac{1}{11} + \frac{1}{11} \right)^2 = \frac{2}{11^2}$$

$\boxed{6.21}$ LÅT $D = \{(x,y) : 1 \le x^2+y^2 \le 2\}$, BERÄKNA

$$I := \iint_D \ln(1+x^2+y^2)\,dx\,dy$$

BYT TILL POLÄRA KOORDINATER

$$\begin{cases} x = r\cdot\cos\varphi \\ y = r\cdot\sin\varphi \end{cases}$$



$1 \le x^2+y^2 \le 2$

$\Updownarrow$

$1 \le r^2 \le 2$

VI FÅR ATT

$$I = \iint_R \ln(1+r^2)\,r\,dr\,d\varphi$$

$$= \int_{r=1}^{\sqrt{2}} r\cdot\ln(1+r^2)\,dr \cdot \int_{\varphi=0}^{2\pi} d\varphi = 2\pi\cdot\left\{\left[\frac{r^2}{2}\ln(1+r^2)\right]_1^{\sqrt{2}} - \frac{1}{2}\int_1^{\sqrt{2}} \frac{r^2\cdot 2r}{1+r^2}\,dr\right\}$$

$$= 2\pi\cdot\left\{\frac{1}{2}\cdot(2\ln 3 - \ln 2) - \int_1^{\sqrt{2}} \frac{r^3}{1+r^2}\,dr\right\}$$

POLYNOMDIVISION GER

$$\begin{array}{r|l} & r \\ \hline 1+r^2 & r^3 \\ & -(r+r^3) \\ \hline & -r \end{array} \implies \frac{r^3}{1+r^2} = r - \frac{r}{1+r^2}$$

SÅ

$$\int_1^{\sqrt{2}} \frac{r^3}{1+r^2}\,dr = \int_1^{\sqrt{2}}\left(r - \frac{r}{1+r^2}\right)dr = \left[\frac{r^2}{2} - \frac{1}{2}\ln(1+r^2)\right]_1^{\sqrt{2}} = \frac{1}{2}\cdot(2 - \ln 3 - 1 + \ln 2)$$

VILKET GER

$$I = 2\pi\cdot\left\{\frac{1}{2}(2\ln 3 - \ln 2) - \frac{1}{2}(2 - \ln 3 - 1 + \ln 2)\right\}$$

$$= \pi\cdot(3\ln 3 - 2\ln 2 - 1) = \pi\cdot\left(\ln\frac{3^3}{2^2} - 1\right) = \pi\cdot\left(\ln\frac{27}{4} - 1\right)$$

$\boxed{6.25}$ LÅT $D = \{(x,y) : x^2+y^2-2x+6y \le 6\}$, BERÄKNA

$$I := \iint_D xy\,dx\,dy$$

KVADRATKOMPLETTERA:

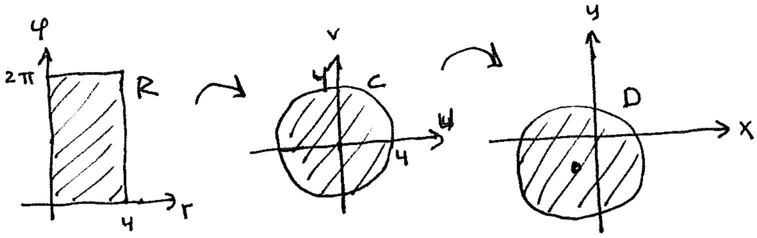
$$x^2-2x+y^2+6y-6 = (x-1)^2-1+(y+3)^2-9-6$$
$$= (x-1)^2+(y+3)^2-16$$

VI SER ATT D ÄR EN CIRKEL MED RADIE 4 OCH CENTRUM I PUNKTEN $(1,-3)$.

GÖR KOORDINATBYTE I TVÅ STEG:

$$\begin{cases} x = u+1 \\ y = v-3 \end{cases}$$

$$\begin{cases} u = r\cdot\cos\varphi \\ v = r\cdot\sin\varphi \end{cases}$$



VI FÅR ATT

$$I = \iint_C (u+1)(v-3)\cdot\left|\frac{d(x,y)}{d(u,v)}\right| du\,dv = \iint_R (r\cos\varphi+1)(r\sin\varphi-3)\left|\frac{d(x,y)}{d(u,v)}\right|\cdot r\cdot dr\,d\varphi$$

$$= \int_{r=0}^{4}\int_{\varphi=0}^{2\pi}(r\cdot\cos\varphi+1)(r\cdot\sin\varphi-3)\cdot r\cdot dr\cdot d\varphi$$

$$= \int_{r=0}^{4}\int_{\varphi=0}^{2\pi} r\cdot\left(r^2\cos\varphi\cdot\sin\varphi-3r\cdot\cos\varphi+r\cdot\sin\varphi-3\right)dr\,d\varphi$$

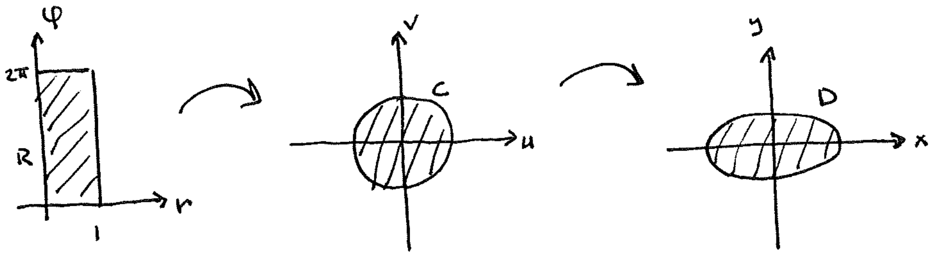
$-3\cdot 2\pi$

$$= \int_{r=0}^{4} r\left(\frac{r^2}{2}\cdot\underbrace{\int_0^{2\pi}\sin 2\varphi\,d\varphi}_{=0}-3r\underbrace{\int_0^{2\pi}\cos\varphi\,d\varphi}_{=0}+r\underbrace{\int_0^{2\pi}\sin\varphi\,d\varphi}_{=0}\right)dr$$

$$= -6\pi\cdot\int_0^4 r\,dr = -6\pi\cdot\left[\frac{r^2}{2}\right]_0^4 = -6\pi\cdot\frac{16}{2} = -48\pi$$

$\boxed{6.26}$ LÅT $D = \{ (x,y) : 2x^2 + 3y^2 \leq 1 \}$, BERÄKNA

$$I := \iint_D (x^2+y^2)\,dx\,dy$$



BYT KOORDINATER I TVÅ STEG

$$\left(\begin{cases} u = \sqrt{2}\cdot x \\ v = \sqrt{3}\cdot y \end{cases} \right) \iff \begin{cases} x = \frac{u}{\sqrt{2}} \\ y = \frac{v}{\sqrt{3}} \end{cases}$$

(AVBILDAR ENHETSSKIVAN PÅ D)

$$\begin{cases} u = r\cdot\cos\varphi \\ v = r\cdot\sin\varphi \end{cases}$$

(AVBILDAR REKTANGELN R PÅ ENHETSSKIVAN)

FUNKTIONALDETERMINANTERNA GES AV

$$\left| \frac{\partial(u,v)}{\partial(r,\varphi)} \right| = r \quad , \quad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\sqrt{6}}$$

ENLIGT SATS 6 (s. 261) FÅR VI

$$I = \iint_C \left(\frac{u^2}{2} + \frac{v^2}{3} \right)\cdot\frac{1}{\sqrt{6}}\,du\,dv = \iint_R \left(\frac{r^2\cdot\cos^2\varphi}{2} + \frac{r^2\cdot\sin^2\varphi}{3} \right)\cdot\frac{1}{\sqrt{6}}\cdot r\,dr\,d\varphi$$

$$= \frac{1}{\sqrt{6}} \int_{r=0}^{1}\int_{\varphi=0}^{2\pi} r^3 \left(\frac{1}{2}\cos^2\varphi + \frac{1}{3}\sin^2\varphi \right) dr\,d\varphi = \frac{1}{\sqrt{6}} \int_{r=0}^{1} r^3\,dr \cdot \left\{ \int_0^{2\pi} \left(\frac{1}{2}\cos^2\varphi + \frac{1}{3}\sin^2\varphi \right) d\varphi \right\}$$

$$= \frac{1}{\sqrt{6}} \cdot \left[\frac{r^4}{4} \right]_0^1 \cdot \left\{ \int_0^{2\pi} \left(\frac{1}{2}\cdot\cos^2\varphi + \frac{1}{3}\left(1-\cos^2\varphi\right) \right) d\varphi \right\}$$

$$= \frac{1}{4\sqrt{6}} \cdot \int_0^{2\pi} \left(\frac{1}{3} + \frac{1}{6}\cos^2\varphi \right) d\varphi = \frac{1}{4\sqrt{6}} \cdot \left(\frac{2\pi}{3} + \frac{1}{6} \underbrace{\int_0^{2\pi} \cos^2\varphi\,d\varphi}_{=\pi} \right)$$

$$= \frac{1}{4\sqrt{6}} \cdot \left(\frac{2\pi}{3} + \frac{\pi}{6} \right) = \frac{5\pi}{4\cdot 6\sqrt{6}} = \frac{5\pi\sqrt{6}}{144}$$

OBS: $\int_0^{2\pi} \cos^2\varphi\,d\varphi = \left[\sin\varphi\cdot\cos\varphi \right]_0^{2\pi} - \int_0^{2\pi} \sin\varphi\cdot(-\sin\varphi)\,d\varphi = \int_0^{2\pi} (1-\cos^2\varphi)\,d\varphi \implies 2\cdot\int_0^{2\pi}\cos^2\varphi\,d\varphi = 2\pi$

$\boxed{6.29}$ LÅT $D = \{ (x,y) : 1 < x^2-y^2 < 4,\ \sqrt{17} < x^2+y^2 < 5,\ x<0,\ y>0 \}$,

BERÄKNA

$$I := \iint_D (x^4 - y^4)\, dx\, dy$$

INFÖR KOORDINATERNA

$$\begin{cases} u = x^2 - y^2 \\ v = x^2 + y^2 \end{cases} \qquad \left| \frac{d(u,v)}{d(x,y)} \right| = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 4xy + 4xy = 8xy$$

(FUNKTIONALDETERMINANTEN ÄR NOLLSKILD OCH $(x,y) \mapsto (u,v)$ ÄR BIJEKTIV PÅ D SÅ DETTA ÄR ETT GILTIGT KOORDINATBYTE.)

BESTÄM NYTT INTEGRATIONSOMRÅDE

$$1 < x^2 - y^2 < 4 \iff 1 < u < 4$$

$$\sqrt{17} < x^2 + y^2 < 5 \iff \sqrt{17} < v < 5$$

$$\left.\begin{matrix} u+v = 2x^2 \\ v-u = 2y^2 \end{matrix}\right\} \iff \begin{cases} x^2 = \frac{1}{2}(u+v) \\ y^2 = \frac{1}{2}(v-u) \end{cases} \overset{(x<0)}{\underset{(y>0)}{\Longrightarrow}} \begin{cases} x = -\sqrt{\frac{u+v}{2}} \\ y = \sqrt{\frac{v-u}{2}} \end{cases}$$

DETTA VISAR ATT AVBILDNINGEN $(x,y) \mapsto (u,v)$ ÄR INVERTERBAR, D.V.S. DEN ÄR BIJEKTIV.

(OM DET INTE HADE GÅTT ATT INVERTERA $(x,y) \mapsto (u,v)$ SÅ HADE INTE KOORDINATBYTET VARIT GILTIGT)

$$\left.\begin{matrix} x < 0 \\ y > 0 \end{matrix}\right\} \Longrightarrow \begin{cases} u + v > 0 & (\text{OK}) \\ v - u > 0 & (\text{OK}, \sqrt{17} > 4) \end{cases}$$

D.V.S. VI VILL INTEGRERA ÖVER $E = \{ (u,v) : 1 < u < 4,\ \sqrt{17} < v < 5 \}$

$$I = \iint_E \left(\tfrac{1}{4}(u+v)^2 - \tfrac{1}{4}(v-u)^2 \right) \cdot \left| \frac{d(x,y)}{d(u,v)} \right| =$$

$$= \frac{1}{4} \iint_E \left(u^2 + 2uv + v^2 - v^2 + 2uv - u^2 \right) \cdot \frac{1}{8xy}\, du\, dv$$

$$= \frac{1}{8} \int_{u=1}^{4} \int_{v=\sqrt{17}}^{5} u \cdot v \cdot \left(\tfrac{1}{2} \sqrt{(u+v)(v-u)} \right)^{-1} du\, dv = \frac{1}{4} \int_{u=1}^{4} \int_{v=\sqrt{17}}^{5} \frac{uv}{\sqrt{v^2 - u^2}}\, du\, dv$$

$$= \frac{1}{4} \int_{v=\sqrt{17}}^{5} v \cdot \left[-\sqrt{v^2 - u^2} \right]_{u=1}^{4} dv = \frac{1}{4} \int_{\sqrt{17}}^{5} v \cdot \left(-\sqrt{-16+v^2} + \sqrt{-1+v^2} \right) dv$$

$$= \frac{1}{4} \left(\frac{1}{3} \left[(v^2-1)^{3/2} - (v^2-16)^{3/2} \right]_{\sqrt{17}}^{5} \right) = \frac{1}{12} \left(24^{3/2} - 9^{3/2} - 16^{3/2} + 1 \right)$$

$$= \frac{1}{12} \cdot \left(6^{3/2} \cdot 2^3 - 27 - 64 + 1 \right) = \frac{6^{3/2} \cdot 8}{2 \cdot 6} - \frac{90}{12} = \frac{1}{2} \cdot \left(8 \cdot 6^{1/2} - 15 \right) = \frac{1}{2} \left(8\sqrt{6} - 15 \right)$$

SAMMANFATTNING — LEKTION 8

BEGREPP:

- VARIABELBYTE (s. 259)
- FUNKTIONALDETERMINANT (s. 140)
  (TOLKNING SOM AREAFÖRSTORING s. 260)

SATSER:

- VARIABELBYTE I DUBBELINTEGRAL, SATS 6 (s. 261)
- FUNKTIONALDETERMINANT FÖR SAMMANSÄTTNING, SATS 1 (s. 141),
- INVERSENS FUNKTIONALDETERMINANT, EKV. (20) (s. 141)