LEKTION 14 4/3/2008 SF1646

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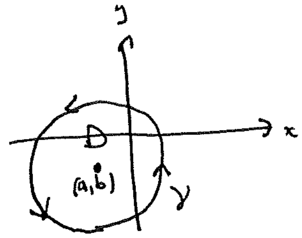
$\boxed{9.10}$ BERÄKNA $I = \int_\gamma y^2 dx + x^2 dy$ DÄR γ ÄR CIRKELN

$(x-a)^2 + (y-b)^2 = r^2$ GENOMLÖPT ETT VARV MOTURS.

ANVÄND GREENS FORMEL:



$$\oint_\gamma P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

DÄR $P(x,y) = y^2$, $Q(x,y) = x^2$, $\partial D = \gamma$

$$I = \iint_D (2x - 2y) \, dx dy =$$

$$= \left\{ \text{BYT KOORD} \quad \begin{array}{l} x = u + a \\ y = v + b \end{array} \right\} =$$

$$= 2 \iint_{u^2+v^2 \le r^2} (u - v + a - b) \, du dv = 2 \int_{\varphi=0}^{2\pi} \int_{\rho=0}^{r} (\rho \cdot \cos\varphi - \rho \cdot \sin\varphi + a - b) \rho \, d\rho \, d\varphi$$

$$= 2 \cdot \left[\int_{\rho=0}^{r} \rho^2 d\rho \underbrace{\int_{\varphi=0}^{2\pi} (\cos\varphi - \sin\varphi) d\varphi}_{=0} + (a-b) \int_{\varphi=0}^{2\pi} d\varphi \cdot \int_{\rho=0}^{r} \rho \, d\rho \right]$$

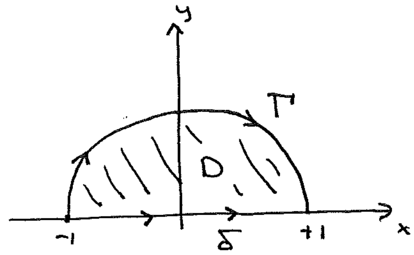
$$= 2(a-b) \cdot 2\pi \cdot \frac{r^2}{2} = 2\pi \cdot (a-b) \cdot r^2$$

9.11 BERÄKNA

$$I := \int_{\Gamma} (x^2 - y + 2\ln(1+y))\,dx + \frac{(1+x)^2}{1+y}\,dy$$

DÄR Γ ÄR ÖVRE HALVAN AV ENHETSCIRKELN MEDURS FRÅN $(-1,0)$ TILL $(1,0)$.

LÅT δ VARA LINJEN FRÅN $(-1,0)$ TILL $(1,0)$,
LÅT D VARA $\{x^2+y^2 \le 1,\ y \ge 0\}$.



GREENS FORMEL GER

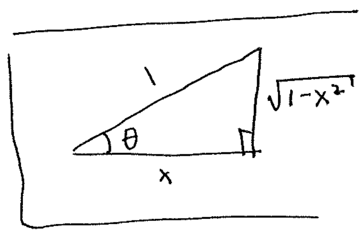
$$\int_{\delta-\Gamma} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

EFTERSOM $\int_{\delta-\Gamma} P\,dx + Q\,dy = \int_{\delta} P\,dx + Q\,dy - \int_{\Gamma} P\,dx + Q\,dy$ FÅR VI ATT

$$I = \int_{\delta} P\,dx + Q\,dy - \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy =: I_1 - I_2$$

PÅ δ ÄR $dy = 0 = y$ SÅ

$$I_1 = \int_{\delta} x^2\,dx = \int_{-1}^{1} x^2\,dx = \frac{1}{3}\left[x^3\right]_{-1}^{1} = \frac{2}{3}$$



FÖR ATT BERÄKNA I_2 DERIVERAR VI FÖRST:

$$\frac{\partial Q}{\partial x} = 2\cdot\frac{1+x}{1+y}, \quad \frac{\partial P}{\partial y} = -1 + \frac{2}{1+y}$$

$$I_2 = \iint_D \left(2\cdot\frac{1+x}{1+y} + 1 - \frac{2}{1+y}\right) dx\,dy = \int_{x=-1}^{1}\left(\int_{y=0}^{\sqrt{1-x^2}}\left(\frac{2x}{1+y} + 1\right) dy\right) dx$$

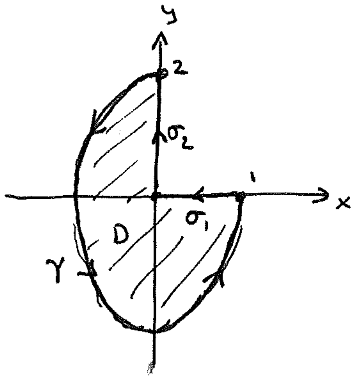
$$= \int_{-1}^{1}\left[2x\cdot\ln(1+y) + y\right]_0^{\sqrt{1-x^2}} dx = \int_{-1}^{1}\Big(\underbrace{2x\cdot\ln\left(1+\sqrt{1-x^2}\right)}_{\text{UDDA}} + \underbrace{\sqrt{1-x^2}}_{\text{JÄMN}}\Big)\,dx$$

$$= 2\int_0^1 \sqrt{1-x^2}\,dx = \left\{\begin{array}{l} x = \cos\theta \\ dx = -\sin\theta\cdot d\theta \end{array}\right\} = -2\cdot\int_{\pi/2}^{0}\sqrt{1-\cos^2\theta}\cdot\sin\theta\,d\theta = 2\int_0^{\pi/2}\sin^2\theta\,d\theta$$

$$= 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$$

$$I = I_1 - I_2 = \frac{2}{3} - \frac{\pi}{2}$$

$\boxed{9.14}$ BERÄKNA ARBETET $W = \int_\gamma \bar{F}\cdot d\bar{r}$ DÅ $\bar{F} = (y^3, x^3) = (P, Q)$ OCH γ ÄR DEN BIT AV ELLIPSEN $x^2 + y^2/4 = 1$ SOM GÅR MOTURS FRÅN $(0,2)$ TILL $(1,0)$.



ENLIGT GREENS FORMEL SÅ GÄLLER

$$\int_{\gamma+\sigma_1+\sigma_2} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

MEN V-L UPPFYLLER ÄVEN

$$\int_{\gamma+\sigma_1+\sigma_2} P\,dx + Q\,dy = \int_\gamma P\,dx + Q\,dy + \int_{\sigma_1} P\,dx + Q\,dy + \int_{\sigma_2} P\,dx + Q\,dy$$

SÅLEDES GÄLLER

$$W := \int_\gamma P\,dx + Q\,dy = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{= I_D} \underbrace{- \int_{\sigma_1} P\,dx + Q\,dy}_{+ I_1} \underbrace{- \int_{\sigma_2} P\,dx + Q\,dy}_{+ I_2}$$

PÅ KURVAN σ_1 GÄLLER $y = 0 = dy$, SÅ

$$I_1 = -\int_{\sigma_1} y^3 dx + x^3 dy = -\int_{\sigma_1} 0\,dx + x^3 \cdot 0 = 0$$

PÅ KURVAN σ_2 GÄLLER $x = 0 = dx$, SÅ

$$I_2 = -\int_{\sigma_2} y^3 dx + x^3 dy = -\int_{\sigma_2} y^3 \cdot 0 + 0 \cdot dy = 0$$

ALLTSÅ FÅR VI ATT

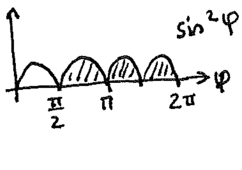
$$W = I_D = \iint_D (3x^2 - 3y^2)\,dx\,dy = \left\{ \begin{array}{l} x = u \\ y = 2v \end{array}, \ \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2 \right\}$$

$$= 3\iint_D (u^2 - 4v^2)\cdot 2\,du\,dv = \left\{ \begin{array}{ll} u = r\cdot\cos\varphi, & 0 \le r \le 1 \\ v = r\cdot\sin\varphi, & \pi/2 \le \varphi \le 2\pi \end{array} \right\}$$

$$= 3\cdot 2\cdot \int_{r=0}^{1} \int_{\varphi=\pi/2}^{2\pi} (r^2\cos^2\varphi - 4r^2\sin^2\varphi)\, r\,d\varphi\,dr = 6 \int_{r=0}^{1} r^3 \int_{\varphi=\pi/2}^{2\pi} (\cos^2\varphi - 4\sin^2\varphi)\,d\varphi\,dr$$

$$= 6\cdot\left[\frac{r^4}{4}\right]_0^1 \cdot \int_{\varphi=\pi/2}^{2\pi} (1 - 5\sin^2\varphi)\,d\varphi = 6\cdot\frac{1}{4}\cdot\left(2\pi - \frac{\pi}{2} - 5\cdot\frac{3\pi}{4}\right) = -\frac{27\pi}{8}$$

$\left(\int_0^{2\pi} \sin^2\varphi\,d\varphi = \pi \right.$, AREAN UNDER ALLA FYRA "KULLAR"

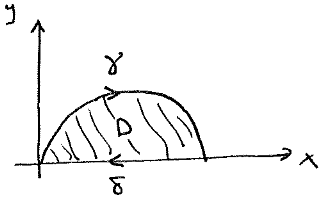


$\left. \int_{\pi/2}^{2\pi} \sin^2\varphi\,d\varphi \text{ ÄR AREAN UNDER DE TRE SISTA "KULLARNA", DVS. } 3\pi/4 \right)$

$\boxed{9.24}$ BERÄKNA AREAN MELLAN X-AXELN OCH CYKLOIDBÅGEN

$$\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases} \qquad 0 \le t \le 2\pi$$

GREENS FORMEL GER



$$\iint_D dxdy = -\int_{-(\gamma+\delta)} y\,dx = \int_{\gamma+\delta} y\,dx =$$

$$= \int_\gamma y\,dx + \int_\delta y\,dx \underset{\substack{\uparrow \\ \left(\substack{y=0 \\ \text{PÅ } \delta}\right)}}{=} \int_\gamma y\,dx = \int_0^{2\pi} y \cdot x'\,dt$$

$$= \int_0^{2\pi} (1-\cos t)\cdot(1-\cos t)\,dt = \int_0^{2\pi} \left(1 - 2\cos t + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} dt - 2\int_0^{2\pi} \cos t\,dt + \int_0^{2\pi} \cos^2 t\,dt = 2\pi - 0 + \pi = 3\pi$$

$$\int_0^{2\pi} \cos^2 t\,dt = \left\{ \begin{aligned} \cos 2t &= \cos^2 t - \sin^2 t \\ &= 2\cos^2 t - 1 \end{aligned} \right\} = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{2\pi} = \frac{1}{2}\cdot\left(2\pi + \frac{1}{2}\sin 4\pi - 0 - \frac{1}{2}\sin 0\right) = \pi$$
