- RIKTNINGSFÄLT
- LINJÄR 1:A ORDN. ODE, LINEARITET, INTEGRERANDE FAKTOR, PRODUKTREGELN FÖR DERIVATA, $f' = 0 \Rightarrow f$ ÄR KONSTANT, $(\log f)' = \frac{f'}{f}$
- SEPARABEL 1:A ORDN. ODE, ~~EXISTENS & ENTYDIGHET ⟺ KONTINUITET~~, DERIVERING LINJÄR, KEDJEREGELN FÖR DERIVERING
- EXISTENS + ENTYDIGHET: I LINJÄRA FALLET FÅR VI <u>GLOBALA</u> LÖSNINGAR, I ICKE LINJÄRA FALLET ÄR LÖSNINGARNA <u>LOKALA</u>;
- ÄVEN OM f EJ KONT. SÅ KAN LÖSN. EXISTERA (EX. 2, S. 71)
- SVÅRT ATT ANGÖRA STÖRSTA LÖSN. OMRÅDE FÖR ICKE LINJ. : MÅSTE LÖSA FÖRST

- TAVLA: 2.1.7, 2.2.4, 2.4.2, 2.4.14
- ÖVA: 2.1.14, 2.2.2, 2.4.4

$$= \frac{df}{dx}(y(x_0)) \cdot \frac{dy}{dx}(x_0)$$

- VARNING: $\frac{d}{dx}\left(f(y)\right)\Big|_{x=x_0} = f'(y)\cdot y'\Big|_{x=x_0}$ (<u>INTE</u> $f'(y)$!)

VAD BETYDER $\frac{d}{dx} f(y)$? FARLIGT!

$\boxed{2.1.7}$

$y' + 2ty = 2te^{-t^2}$

a) DRAW DIRECTION FIELD

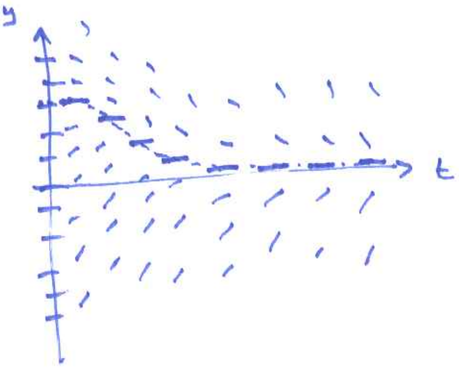
$$y' = 2t\cdot\left(e^{-t^2} - y\right) =: f(t,y)$$

(LINEAR ODE SINCE f IS AFFINE IN y:
$f(t, y_1 - y_2) = f(t,y_1) - f(t,y_2)$)

THE SIGN OF f WILL TELL US A LOT ABOUT DIRECTION FIELD, SO LOOK FOR ZEROES OF f:

$$f(t,y) = 0 \iff \begin{cases} t = 0 \\ \text{or} \\ y = e^{-t^2} \end{cases}$$

SO ON THE LINE $t=0$ AND CURVE $y = e^{-t^2}$ THE SLOPE IS 0. IF $y < e^{-t^2}$ THE SLOPE IS POSITIVE, AND IF $y > e^{-t^2}$ THE SLOPE IS NEGATIVE. ALSO, THE $2t$ TERM GROWS SO THE SLOPE IS FARTHER FROM 0 AWAY FROM $t=0$. SAME FOR $|y| \to \infty$.



b) HOW DO SOLUTIONS BEHAVE FOR LARGE t JUDGING FROM DIRECTION FIELD?

SEEMS THEY ALL $\to 0$, SINCE $e^{-t^2} \to 0$

c) SOLVE EQN. AND DESCRIBE HOW $y(t)$ BEHAVES AS $t \to \infty$.

LINEAR ODE. FIND INTEGRATING FACTOR

$$\frac{d}{dt}(h\cdot y) = h\cdot y' + h'y = h\cdot y' + h\cdot 2ty = h\cdot 2te^{-t^2}$$

$$\Rightarrow h' = 2t\cdot h \quad \Leftarrow h = e^{t^2}$$

$$\frac{d}{dt}\left(e^{t^2}\cdot y\right) = 2t\cdot e^{-t^2}\cdot e^{t^2} = 2t \quad \Rightarrow e^{t^2}\cdot y = t^2 + c$$

$$\Rightarrow y = e^{-t^2}\cdot\left(t^2 + c\right)$$

NB. $t^2 e^{-t^2} \to 0$ SO SOLN. $y \to 0$ AS $t \to \infty$.

$\boxed{2.1.13}$ FIND THE SOLUTION OF

$$y' - y = 4te^{2t}, \qquad y(0) = 1$$

LOOK FOR INTEGRATING FACTOR h S.T.

$$\frac{d}{dt}(h \cdot y) = h'y + h \cdot y' = h \cdot y' - y \cdot h$$

$$\Rightarrow \quad h' = -h$$

TAKE E.G. $h(t) = e^{-t}$ (THEN $h' = -e^{-t} = -h$).

HENCE

$$\frac{d}{dt}\left(e^{-t} \cdot y(t)\right) = e^{-t} \cdot 4te^{2t} = 4te^{t}$$

$$e^{-t} \cdot y(t) = \int_{t_0}^{t} 4se^{s}\,ds + C$$

FOR SOME CONSTANT C, AND ARBITRARY t_0.

$$4\int_{t_0}^{t} se^{s}\,ds = 4 \cdot \left(\left[e^{s} \cdot s\right]_{t_0}^{t} - \int_{t_0}^{t} e^{s}\,ds\right)$$

$$= 4 \cdot \left(t \cdot e^{t} - e^{t_0} \cdot t_0 - e^{t} + e^{t_0}\right)$$

CHOOSE E.G. $t_0 = 0$. THEN

$$y(t) = e^{t} \cdot 4 \cdot \left(te^{t} - e^{t} + 1\right) + C \cdot e^{t}$$

INITIAL VALUE COND. $y(0) = 1$ GIVES

$$1 = y(0) = 4 \cdot \left(0 \cdot e^{0} - e^{0} + 1\right) + C \cdot e^{0} = C$$

ANSWER: $y(t) = 4e^{t} \cdot \left(1 + (t-1)e^{t}\right) + e^{t}$

$$= 5e^{t} + 4(t-1)e^{2t}$$

CHECK:

$$y' = 5e^{t} + 8(t-1)e^{2t} + 4e^{2t}$$

$$-y = -5e^{t} - 4(t-1)e^{2t}$$

$$4(t-1)e^{2t} + 4e^{2t} = 4te^{2t} \quad \text{OK}$$

$\boxed{2.1.14}$ $y' + 3y = te^{-3t}, \quad y(1) = 0$

$$h \cdot y' + 3h \cdot y = h \cdot te^{-3t}$$

$$\Rightarrow \quad h' = 3h \quad \Leftarrow h(t) = e^{3t}$$

$$\frac{d}{dt}(h \cdot y) = hy' + h'y$$

$$\frac{d}{dt}\left(e^{3t} \cdot y\right) = t \quad \Rightarrow \quad e^{3t} \cdot y = \int_0^t s\,ds + C = \frac{t^2}{2} + C$$

$$y(t) = e^{-3t} \cdot \left(\frac{t^2}{2} + C\right)$$

$$0 = y(1) = e^{-3} \cdot \left(\frac{1}{2} + C\right) \quad \Rightarrow \quad C = -\frac{1}{2}$$

$$y(t) = e^{-3t} \cdot \left(\frac{t^2}{2} - \frac{1}{2}\right) = \boxed{\frac{e^{-3t}}{2} \cdot (t^2 - 1)}$$

$$y' + 3y - te^{-3t} = \underbrace{-\frac{3e^{-3t}}{2}(t^2-1)}_{/} + \underbrace{\frac{e^{-3t}}{2} \cdot 2t}_{//} + \underbrace{\frac{3e^{-3t}}{2}(t^2-1)}_{/} - \underbrace{te^{-3t}}_{//} = 0$$

ok

$\boxed{2.1.33}$ $a,\lambda>0$, $b\in\mathbb{R}$, $y'+ay=be^{-\lambda t} \Rightarrow y(t)\to 0$, $t\to\infty$

P.B. $\frac{d}{dt}(e^{+at}\cdot y) = e^{+at}y' + ae^{+at}\cdot y = e^{+at}(y'+ay) = be^{(a-\lambda)t}$

$$\Rightarrow \quad e^{at}\cdot y = b\cdot\int_0^t e^{(a-\lambda)s}\,ds + C = \begin{cases} \frac{be^{(a-\lambda)t}}{a-\lambda} + C, & a\neq\lambda \\ b\cdot t + C, & a=\lambda \end{cases}$$

$$y(t) = \begin{cases} e^{-at}\cdot\left(\frac{be^{(a-\lambda)t}}{a-\lambda} + C\right) = \frac{be^{-\lambda t}}{a-\lambda} + Ce^{-at}, & a\neq\lambda \\ e^{-at}(b\cdot t + C), & a=\lambda \end{cases}$$

IN BOTH CASES $y(t)\to 0$ AS $t\to\infty$, SINCE $a,\lambda>0$

$\boxed{2.1.35}$ ~~FIND 1ST ORDER LINEAR ODE. S.T. ALL SOLN. ASYMPTOTIC TO $y=2-t$ AS $t\to\infty$~~

~~P.B. $y=2-t$ SATISFIES~~ $y'=-1$

$\boxed{2.2.2}$ SOLVE $y' = \dfrac{3x^2}{y(1+x^3)}$

NOTE: RHS IS <u>NOT</u> CONT. AT $y=0$ AND AT $x=-1$.

SEPARABLE EQN.

$$(*) \qquad \int y\,dy = \int \frac{3x^2}{1+x^3}\,dx + c$$

IF $x>-1$: $(1+x^3)>0$, SO $\log(1+x^3)$ IS DEF. AND

$$\frac{d}{dx}\{\log(1+x^3)\} = 3x^2/(1+x^3).$$

IF $x<-1$: $1+x^3<0$, SO $\log-(1+x^3)$ IS DEF. AND

$$\frac{d}{dx}\{\log-(1+x^3)\} = -3x^2/(-(1+x^3)) = 3x^2/(1+x^3).$$

HENCE

$$\int \frac{3x^2}{1+x^3}\,dx = \log|1+x^3|, \qquad x \neq -1$$

THUS $(*)$ IMPLIES

$$\boxed{\frac{y^2}{2} = \log|1+x^3| + c}$$

CHECK:

$$\left.\begin{aligned} \frac{d}{dx}\left(\frac{y^2}{2}\right) &= \frac{2y}{2}\cdot y' = y\cdot y' \\ \frac{d}{dx}\{\log|1+x^3|+c\} &= \frac{3x^2}{1+x^3} \end{aligned}\right\} \Rightarrow y' = \frac{3x^2}{y(1+x^3)} \quad \underline{\underline{\text{OK}}}$$

2.2.4

SOLVE $y' = \frac{3x^2-1}{4+2y}$, NB. NOT CNT. FOR $y=-2$

SEPARABLE EQN.

$$\int (4+2y)\,dy = \int (3x^2-1)\,dx + c$$

$$4y + y^2 = x^3 - x + c$$

$$\boxed{y(y+4) = x(x^2-1) + c}$$

CHECK:

$$\frac{d}{dx}\left(y(y+4)\right) = y'(y+4) + yy' = (2y+4)y'$$

$$\frac{d}{dx}\left(x(x^2-1)+c\right) = x^2-1 + x\cdot 2x = 3x^2-1$$

$$\Rightarrow y' = \frac{3x^2-1}{2y+4} \quad \text{QED}$$

ALT.)

$$\underbrace{3x^2-1}_{\frac{d}{dx}\{x^3-x\}} - \underbrace{(4+2y)y'}_{\frac{d}{dx}\{4y+y^2\}} = \underbrace{0}_{\frac{d}{dx}\{c\}}$$

USE:

$$f'(x) + g'(y)y' = \frac{d}{dx}\left(f(x)+g(y)\right)$$

(CHAIN RULE + $\frac{d}{dx}$ IS LINEAR)

$$\frac{d}{dx}\{x^3-x-4y-y^2\} = \frac{d}{dx}\{c\}$$

$$\boxed{x^3-x-4y-y^2 = c}$$

$\boxed{2.4.2}$ BESTÄM INTERVALL DÄR BVP HAR LÖSNING

$$t(t-5)y' + y = 0 \quad , \qquad y(3) = 1$$

KAN SKRIVAS PÅ STANDARDFORM

$$y' + \underbrace{\frac{1}{t(t-5)}}_{p(t)} \cdot y = 0 \quad , \qquad t \notin \{0,5\}.$$

DET ÄR EN LINJÄR ODE SÅ LÖSN. EXIST. DÄR $p(t)$ ÄR KONT.

D.V.S. DÅ $t \notin \{0,5\}$. VI HAR EN VILLKORET $y(3) = 1$

SÅ LÖSN. TILL BVP EXIST. DÅ $0 < t < 5$.

$\boxed{2.4.4}$ IBID FÖR

$$(16 - t^2)y' + 2ty = 3t^2 \quad , \qquad y(-5) = 1$$

STANDARDFORM:

$$y' + \frac{2t}{16-t^2} \cdot y = \frac{3t^2}{16-t^2} \quad , \qquad t \neq \pm 4$$

LINJÄR ODE, LÖSN. EXIST. FÖR $t < -4$.

$\boxed{2.4.14}$ LÖS BVP OCH BESTÄM HUR LÖSN. DEFINITIONSOMRÅDE BEROR PÅ y_0.

$$y' = 4ty^2, \qquad y(0) = y_0.$$

OBS: HL KONT.
$\frac{\partial}{\partial y}$ HL KONT.
ENTYDIG LÖSN. EXIST. LOKALT
MEN OMÖJLIGT ATT AVGÖRA GLOBALA INTERVALLET DÄR LÖSN. FINNS

SEPARABEL ODE

$$\int \frac{dy}{y^2} = \int 4t\,dt + c$$

$$-\frac{1}{y} = 2t^2 + c$$

B.V. $y(0) = y_0$ GER

$$-\frac{1}{y_0} = 2\cdot 0^2 + c \implies c = -\frac{1}{y_0} \quad \text{OM } y_0 \neq 0$$

DÅ $y_0 = 0$ SAKNAR OVANST. EKV. LÖSN. MEN LÄGG MÄRKE TILL ATT BVP HAR LÖSN. $y = 0$ DÅ $y_0 = 0$ (DEF. FÖR ALLA t).

ANTAG $y_0 \neq 0$, DÅ

$$y = \left(-2t^2 + \frac{1}{y_0}\right)^{-1}, \qquad -2t^2 + \frac{1}{y_0} \neq 0 \quad \text{DVS} \quad t \neq \pm\frac{1}{\sqrt{2y_0}}$$

OM $y_0 > 0$

OM $y_0 < 0$ ÄR y DEF. ÖVERALLT TY $-2t^2 + \frac{1}{y_0} \neq 0$.

OM $y_0 > 0$ ÄR y DEF. FÖR $-\frac{1}{\sqrt{2y_0}} < t < \frac{1}{\sqrt{2y_0}}$ (OBS $t_0 = 0$).