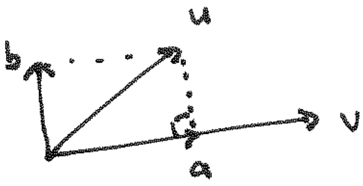
CAUCHY – SCHWARTZ:

$$|\langle u,v\rangle| \leq \|u\|\cdot\|v\|$$

$$\left(|\langle u,v\rangle|^2 \leq \langle u,u\rangle \cdot \langle v,v\rangle \right)$$

BEVIS: IDEN ÄR ATT BETRAKTA PROJICERINGEN AV u PÅ v



$b = u - a$

LÅT $\hat{v} = \dfrac{v}{\|v\|}$, D.V.S. NORMERAD VERSION AV v.

DÅ

$$a = \langle u, \hat{v}\rangle \hat{v} = \left\langle u, \frac{v}{\|v\|}\right\rangle \frac{v}{\|v\|} = \frac{\langle u,v\rangle}{\|v\|^2}\cdot v$$

$$= \underbrace{\frac{\langle u,v\rangle}{\langle v,v\rangle}}_{\alpha}\cdot v =: \alpha\cdot v$$

OBS ATT

$$\langle b,v\rangle = \langle u-\alpha v, v\rangle = \langle u,v\rangle - \alpha\langle v,v\rangle = \langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle}\langle v,v\rangle$$

$$= 0$$

D.V.S. $|\langle u,v\rangle| = |\langle a+b, v\rangle| = |\langle a,v\rangle + \langle b,v\rangle| = |\langle a,v\rangle|$, SÅ ENDAST PROJICERINGEN AV u PÅ v BIDRAR TILL $\langle u,v\rangle$!

BETRAKTA LÄNGDEN PÅ b, D.V.S. $\|b\|$:

$$0 \leq \|b\|^2 = \langle b,b\rangle = \langle u-\alpha v, u-\alpha v\rangle = \langle u, u-\alpha v\rangle - \alpha\langle v, u-\alpha v\rangle$$

$$= \langle u,u\rangle - \bar{\alpha}\langle u,v\rangle - \alpha\langle v,u\rangle + \alpha\bar{\alpha}\langle v,v\rangle$$

$$= \langle u,u\rangle + |\alpha|^2\langle v,v\rangle - \left(\underbrace{\bar{\alpha}\langle u,v\rangle}_{z} + \underbrace{\overline{\bar{\alpha}\langle u,v\rangle}}_{\bar{z}}\right)$$

$$= \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^4}\cdot\|v\|^2 - 2\cdot \mathrm{Re}\{\bar{\alpha}\cdot\langle u,v\rangle\}$$

$$= \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} - 2\cdot|\alpha|\cdot|\langle u,v\rangle| = \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} - 2\cdot\frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

$$\Rightarrow |\langle u,v\rangle|^2 \leq \|u\|^2\cdot\|v\|^2 \qquad \square$$

(OBS! LIKHET PRECIS DÅ $\|b\|=0$, D.V.S. DÅ u,v ÄR LINJÄRT BEROENDE)

$\boxed{5.2}$ $$\langle f, g \rangle = \int_{-1}^{1} f(x)\overline{g(x)}\,dx$$

LÅT $f(x) = x+2$, $g(x) = ix + x^2$. BERÄKNA $\langle f, g \rangle$.

$$\langle f, g \rangle = \int_{-1}^{1} (x+2)\overline{(ix+x^2)}\,dx = \int_{-1}^{1} \Big(\underbrace{-2ix}_{\text{UDDA}} + \underbrace{(2+i)x^2}_{\text{JÄMN}} + \underbrace{x^3}_{\text{UDDA}}\Big)\,dx$$

$$= 2\int_{0}^{1} (-2ix + x^3)\,dx = 2\cdot\left(-2i\left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^4}{4}\right]_0^1\right)$$

$$= 2\cdot\left(-2i\cdot\frac{1}{2} + \frac{1}{4}\right) = \underline{\underline{-2i + \frac{1}{2}}}$$

5.5 ORTOGONALISERA FÖLJANDE VEKTORER

a) $u = (1,2,3)$, $v = (3,1,4)$, $w = (2,1,1)$ (i $\mathbb{C}^3$).

$$\begin{cases} x_1 = u \\ x_2 = v - \dfrac{\langle v, u\rangle}{\langle u,u\rangle}\cdot u \\ x_3 = w - \dfrac{\langle w, u\rangle}{\langle u,u\rangle}\cdot u - \dfrac{\langle w, x_2\rangle}{\langle x_2, x_2\rangle}\cdot x_2 \end{cases}$$

(GRAM-SCHMIDT)

$\langle u,u\rangle = 1^2+2^2+3^2 = 14$ $\langle v,v\rangle = 3^2+1^2+4^2 = 26$

$\langle v,u\rangle = 3\cdot 1+1\cdot 2+4\cdot 3 = 17$

$\langle w,u\rangle = 2\cdot 1+1\cdot 2+1\cdot 3 = 7$ $\langle w,v\rangle = 2\cdot 3+1\cdot 1+1\cdot 4 = 11$

$$x_2 = (3,1,4) - \frac{17}{14}(1,2,3) = \left(3-\frac{17}{14},\ 1-\frac{17}{7},\ 4-\frac{17}{14}\cdot 3\right)$$

$$\begin{cases} x_1 = (1,2,3) \\ x_2 = \left(\frac{25}{14}, -\frac{10}{7}, \frac{5}{14}\right) \end{cases}$$

$\langle w, x_2\rangle = \frac{5}{2}$

$\langle x_2, x_2\rangle = \frac{75}{14}$

$$x_3 = (2,1,1) - \frac{7}{14}(1,2,3) - \frac{5/2}{75/14}\left(\frac{25}{14}, -\frac{10}{7}, \frac{5}{14}\right) = \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

SVAR:
$$\begin{cases} (1,2,3) \\ \left(\frac{25}{14}, -\frac{10}{7}, \frac{5}{14}\right) \\ \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right) \end{cases}$$

$\boxed{5.5}$

b) $f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2 \quad , \quad C(-1,1)$

$$g_1(x) = f_1(x)$$

$$g_2(x) = f_2(x) - \frac{\langle f_2, g_1 \rangle}{\langle g_1, g_1 \rangle} \cdot g_1(x)$$

$$g_3(x) = f_3(x) - \frac{\langle f_3, g_1 \rangle}{\langle g_1, g_1 \rangle} \cdot g_1(x) - \frac{\langle f_3, g_2 \rangle}{\langle g_2, g_2 \rangle} \cdot g_2(x)$$

$$\left.\begin{aligned} \langle g_1, g_1 \rangle &= \int_{-1}^{1} 1 \cdot 1 \, dx = 2 \\ \langle f_2, g_1 \rangle &= \int_{-1}^{1} x \cdot 1 \, dx = 0 \end{aligned}\right\} \Rightarrow g_2 = f_2$$

$$\langle f_3, g_1 \rangle = \int_{-1}^{1} x^2 \cdot 1 \, dx = 2 \left[\frac{x^3}{3} \right]_0^1 = \frac{2}{3}$$

$$\langle f_3, g_2 \rangle = \int_{-1}^{1} x^2 \cdot x \, dx = 0$$

$$g_3(x) = x^2 - \frac{2/3}{2} \cdot 1 = x^2 - \frac{1}{3}$$

SVAR: $\left\{ 1, x, x^2 - \frac{1}{3} \right\}$

5.7 BESTÄM ETT POLYNOM $p(x)$ AV GRAD ≤ 1 SOM MINIMERAR

$$\int_0^2 |e^x - p(x)|\,dx$$

BETRAKTA RUMMET $C(0,2)$. OVANSTÅENDE PROBLEM KAN OMFORMULERAS SOM ATT MINIMERA

$$\|e^x - p\| \quad \text{ÖVER} \quad p \in V = \{a + bx : a, b \in \mathbb{C}\}.$$

(ENL. SATS 5.3)

V ÄR DELRUM I $C(0,2)$ SÅ SVARET GES AV

$$p(x) = \langle e^x, \varphi_1 \rangle \varphi_1 + \langle e^x, \varphi_2 \rangle \varphi_2$$

DÄR $\{\varphi_1, \varphi_2\}$ ÄR EN ON-BAS FÖR V.

VI VET ATT $\{1, x\}$ ÄR EN BAS FÖR V SÅ ANVÄND GRAM-SCHMIDT FÖR ATT HITTA ON-BAS

$$\langle 1, 1 \rangle = \int_0^2 1 \cdot 1\,dx = 2 \implies \varphi_1(x) = \frac{1}{\sqrt{2}} \quad \text{HAR NORM 1.}$$

$$\tilde{\varphi}_2(x) = x - \langle x, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}} = x - \frac{1}{2}\int_0^2 x\,dx = x - \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$$

$$= x - 1$$

$$\langle \tilde{\varphi}_2, \tilde{\varphi}_2 \rangle = \int_0^2 (x-1)^2\,dx = \int_{-1}^{1} t^2\,dt = 2 \cdot \left[\frac{t^3}{3}\right]_0^1 = \frac{2}{3}$$

$$\implies \varphi_2(x) = \frac{x-1}{\sqrt{2/3}} = \sqrt{\frac{3}{2}} \cdot (x-1) \quad \text{HAR NORM 1.}$$

ON-BAS FÖR V: $\Big\{ \underbrace{\tfrac{1}{\sqrt{2}}}_{\varphi_1}, \underbrace{\sqrt{\tfrac{3}{2}} \cdot (x-1)}_{\varphi_2} \Big\}$

$$\langle e^x, \varphi_1 \rangle = \int_0^2 \frac{e^x}{\sqrt{2}}\,dx = \frac{1}{\sqrt{2}}\left[e^x\right]_0^2 = \frac{e^2 - 1}{\sqrt{2}}$$

$$\langle e^x, \varphi_2 \rangle = \int_0^2 e^x \cdot \sqrt{\tfrac{3}{2}} \cdot (x-1)\,dx = \sqrt{\tfrac{3}{2}} \cdot \left(\left[e^x \cdot (x-1)\right]_0^2 - \int_0^2 e^x\,dx \right)$$

$$= \sqrt{\tfrac{3}{2}} \cdot \left(e^2 + 1 - \left[e^x\right]_0^2 \right) = \sqrt{\tfrac{3}{2}} \left(e^2 + 1 - e^2 + 1 \right) = 2\sqrt{\tfrac{3}{2}} = \sqrt{6}$$

<u>SVAR</u>: $p(x) = \dfrac{e^2 - 1}{\sqrt{2}} \cdot \varphi_1(x) + \sqrt{6} \cdot \varphi_2(x)$