SCHWARTZ CLASS:

$$\mathcal{S} = \{ \varphi \in C^{\infty}(\mathbb{R}) : \overset{\exists C_{n,k}}{\vee} (1+|x|)^n \cdot |\varphi^{(k)}(x)| \leq C_{n,k} ,\ \forall x,\ \forall n,k \geq 0 \}$$

TEMPERED FUNCTIONS:

$$\mathcal{T} = \{ \chi : \exists m, C : |\chi(x)| \leq C \cdot (1+|x|)^m \}$$

MULTIPLICATOR FUNCTIONS:

$$\mathcal{M} = \{ \chi \in C^{\infty}(\mathbb{R}) : \chi^{(k)} \in \mathcal{T} ,\ \forall k \geq 0 \}$$

CONVERGENCE IN $\mathcal{S}$:

$$\varphi_n \xrightarrow{\mathcal{S}} \varphi \iff \lim_{j \to \infty} \sup_x (1+|x|)^n \cdot |\varphi_j^{(k)}(x) - \varphi^{(k)}(x)| = 0$$

DISTRIBUTION (TEMPERED): $\mathcal{S}'$

$f : \mathcal{S} \to \mathbb{C}$:
- $f[a_1\varphi_1 + a_2\varphi_2] = a_1 f[\varphi_1] + a_2 f[\varphi_2]$, $\varphi_k \in \mathcal{S}$, $a_k \in \mathbb{C}$
- $\varphi_j \xrightarrow{\mathcal{S}} \varphi \Rightarrow f[\varphi_j] \to f[\varphi]$

EX: (1) $T_f[\varphi] = \int_{-\infty}^{\infty} f(x)\varphi(x)dx$ $\qquad$ f LOC. L^1, $f \in \mathcal{T}$

(2) $\delta[\varphi] = \varphi(0)$ $\qquad$ (DIRAC)

(3) $H[\varphi] = \int_0^{\infty} \varphi(x)dx$ $\qquad$ (HEAVISIDE)

(4) $P.V \frac{1}{x}[\varphi] = \lim_{\varepsilon \searrow 0} \int_{|x|>\varepsilon} \frac{\varphi(x)}{x}dx$ $\qquad$ $\left(= \lim_{x \searrow 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x) - \varphi(-x)}{x} dx \right)$

EGENSKAPER:

(1) $f = g \in \mathcal{S}' \iff f[\varphi] = g[\varphi]$, $\forall \varphi \in \mathcal{S}$

(2) $f + g = h \iff h[\varphi] = f[\varphi] + g[\varphi]$, $\qquad$ $g = c \cdot f$ ($c \in \mathbb{C}$) $\iff g[\varphi] = c \cdot f[\varphi]$

(3) $f'[\varphi] = -f[\varphi]$

SATS 8.1: $f \in \mathcal{S}'$ $\quad$ $f' = 0 \iff f$ IS A CONSTANT FUNCTION

8.1) OBS: $\varphi \in \mathcal{S} \Rightarrow \varphi' \in \mathcal{S}$

GÄLLER $\varphi' \in \mathcal{S} \Rightarrow \varphi \in \mathcal{S}$??

$\mathcal{S}$: $(1+|x|)^n \cdot |\varphi^{(k)}| \leq C_{n,k}$, $\forall n,k \geq 0$

$\varphi(x) = \frac{1}{|x|}$, $|x| \geq 1$ $\qquad$ $\varphi^{(n)}(x) = (-1)^n \cdot x^{-(1+n)}$, $x>0$

$\int_1^x \varphi(t)\,dt = \log x$ $\qquad$ NOT IN $\mathcal{S}$! SINCE $(1+|x|)^{n+2} |\varphi^{(n)}| \to \infty$

$$\int_0^x e^{-t}\,dt = \left[-e^{-t}\right]_0^x = 1-e^{-x}$$

$\boxed{\varphi(x) = e^{-t}}$, $t>0$ $\qquad$ $\varphi \in \mathcal{S}$?

$\Phi(x) = \int_0^x \varphi(t)\,dt = 1-e^{-x}$ $\qquad$ $(1+|x|)\cdot\Phi(x) \longrightarrow \infty$

$\Phi \notin \mathcal{S}$!

$\varphi^{(n)}(x) = (-1)^n \cdot \varphi(x)$ $\qquad$ SVAR: NEJ

8.20)

- $e^{-x^2} \in \mathcal{S}, \mathcal{M}$
- $e^{-|x|^5} \notin \mathcal{S}, \mathcal{M}$ (EJ C^∞, $f^{(5)}$ EJ KONT.)
- $\sin(x^2) \in \mathcal{M} \notin \mathcal{S}$
- $x^n \in \mathcal{M} \notin \mathcal{S}$
- $(1+x^2)^{-1} \in \mathcal{M} \notin \mathcal{S}$

TEMPERED DISTR. $\mathcal{S}'$ DUAL OF $\mathcal{S}$

VILKA AV FÖLJANDE DEF. EN TEMPERERAD DISTRIBUTION?

8.3)

(a) $f[\varphi] = \int (2x^2+3)\varphi''(x)\,dx$

(b) $= \int e^x \varphi(x)\,dx$

(c) $= \varphi(0)^2$

(a) OK, $\varphi'' \in \mathcal{S}$ $\quad$ $(2x^2+3)$ POLY. GROWTH
(KAN SKRIVAS $\int (2x^2+3)'' \varphi(x)\,dx$)

(b) NO, e^x GROWS TOO QUICKLY, E.G. $\varphi(x) = e^{-x}$ FOR $x>k$

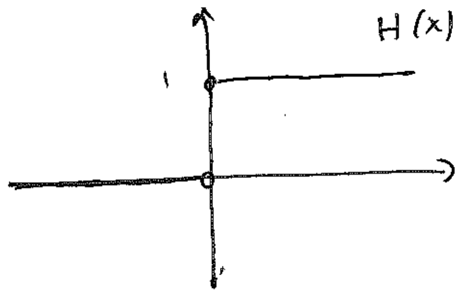
THEN $\int_k^\infty e^x \cdot \varphi(x)\,dx$ DIV.

(c) NO, NOT LINEAR

$$\neq \begin{cases} f[\varphi_1+\varphi_2] = (\varphi_1(0)+\varphi_2(0))^2 = \varphi_1(0)^2 + \varphi_2(0)^2 + 2\varphi_1(0)\varphi_2(0) \\ f[\varphi_1]+f[\varphi_2] = \varphi_1(0)^2 + \varphi_2(0)^2 \end{cases}$$

(2) $H'[\varphi] = -H[\varphi'] = -\int_0^\infty \varphi'(x)\,dx = -[\varphi(x)]_0^\infty = \varphi(0)$

$\Rightarrow \quad H' = \delta$



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(1) $T_f[\varphi] = \int_{-\infty}^{\infty} f(x)\,\varphi(x)\,dx \quad = f[\varphi]$

$T_{f'}[\varphi] = \int_{\mathbb{R}} f'(x)\,\varphi(x)\,dx = [f(x)\varphi(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx$

$= -\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx = -T_f[\varphi']$

$\rightsquigarrow$ DEFINIERA DISTRIBUTIONSDERIVATA ENLIGT $f'[\varphi] = -f[\varphi']$

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(3) $\delta'[\varphi] = -\delta[\varphi'] = -\varphi'(0)$

$\delta''[\varphi] = -\delta'[\varphi'] = \delta[\varphi''] = \varphi''(0)$

$\vdots$

$\delta^{(n)}[\varphi] = (-1)^n \varphi^{(n)}(0)$

8.4) VISA ATT $x^2\delta''' = 6\delta'$

$$(x^2\delta''')[\varphi] = \int x^2\delta'''(x)\,\varphi(x)\,dx$$

$$= -\int \delta''(x)\cdot\left(2x\varphi(x) + x^2\varphi'(x)\right)dx$$

$$= \int \delta'(x)\cdot\left(2\varphi(x) + 2x\varphi'(x) + 2x\varphi'(x) + x^2\varphi''(x)\right)dx$$

$$= -\int \delta(x)\cdot\left(2\varphi'(x) + 2\varphi'(x) + 2x\varphi''(x) + 2\varphi'(x) + 2x\varphi''(x) + 2x\varphi''(x) + x^2\varphi'''(x)\right)dx$$

$$= -\left(\overset{6}{\cancel{4}}\,\varphi'(0)\right) = -6\cdot\varphi'(0)$$

$$(6\delta')[\varphi] = \int 6\delta'(x)\varphi(x)\,dx = -6\int\delta(x)\,\varphi'(x)\,dx = -6\varphi'(0)$$

$$(x^2\delta''')[\varphi] = \delta'''[\underbrace{x^2\varphi}_{\psi}] = \delta'[\psi'']$$

$$\psi''(x) = \left(2x\varphi(x) + x^2\varphi'(x)\right)' = 2\varphi(x) + 2x\varphi'(x) + 2x\varphi'(x) + x^2\varphi''(x)$$

SATS 8.2 $\quad \mathcal{S} \ni \varphi \longmapsto \hat{\varphi} \in \mathcal{S}$ ÄR KONT., BIJEKTIV

INVERS GES AV

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\varphi}(\omega) e^{i\omega x} d\omega$$

KAN DEFINIERA F-TRANSFORM FÖR $f \in \mathcal{S}'$ ENLIGT

$$\hat{f}[\varphi] = f[\hat{\varphi}]$$

EX: $\quad f(x) = c, \ \forall x \quad c$ KONST.

$$\hat{f}[\varphi] = f[\hat{\varphi}] = \int f(x)\, \hat{\varphi}(x) dx = \int c \cdot \hat{\varphi}(x) dx$$

$$= c \cdot 2\pi \cdot \left(\frac{1}{2\pi} \int \hat{\varphi}(x) \cdot e^{ix \cdot 0} dx \right) = c \cdot 2\pi \cdot \varphi(0)$$

$$= c \cdot 2\pi \cdot \delta[\varphi]$$

$\Longrightarrow \quad \hat{c} = 2\pi c \cdot \delta \quad$ (I DISTRIBUTIONSMENING)

OBS! $\hat{f}$ EXIST. EJ I VANLIG MENING

EFTERSOM $\int |f(x)| dx = \infty$

EX: $\quad f(x) = x \quad$ OBS $\ f \notin L^1 \ $ TY $\int |f(x)| dx = \infty$

SÅ VANLIGA F-TRANSF. EJ DEFINIERAD!

$$\hat{f}[\varphi] = f[\hat{\varphi}] = \int f(x)\, \hat{\varphi}(x) dx = \int x \cdot \hat{\varphi}(x) dx$$

$$= \left\{ \text{OBS: } \widehat{\varphi'}(\omega) = i\omega \cdot \hat{\varphi}(\omega) \right\} = \int (-i) \widehat{\varphi'}(x) dx$$

$$= -i 2\pi \cdot \left(\frac{1}{2\pi} \int \widehat{\varphi'}(x) e^{ix \cdot 0} dx \right) = -2\pi i \cdot \varphi'(0)$$

$$= 2\pi i \cdot \delta'[\varphi]$$

$\Longrightarrow \quad \hat{x} = 2\pi i \delta' \qquad \left(\widehat{x^n} = 2\pi i^n \delta^{(n)} \right)$

KAN ANV. FÖR ATT F-TRANSF. POLYNOM

LÖSNING TILL DIRICHLETS PROBLEM PÅ ENHETSSKIVAN:

$$u(r,\theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{in\theta}$$

$$\begin{cases} \Delta u = 0 & \text{PÅ } \mathbb{D} \\ u(e^{i\theta}) = g(\theta) \end{cases}$$

$$c_n = \frac{1}{2\pi} \int_{\mathbb{T}} g(\theta) e^{-in\theta} \, d\theta$$

$$\left(\Delta u = u_{xx} + u_{yy} = r\frac{\partial}{\partial r}\left(r \cdot \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} \right)$$

$\left(\lim_{r \nearrow 1} u(r,\theta) = g(\theta), \ \forall \theta \text{ DÄR } g \text{ ÄR KONT.} \right)$

6.12

$$g(\theta) = 2 + \cos 3\theta + \sin 4\theta$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (2 + \cos 3\theta + \sin 4\theta)(\cos n\theta - i \sin n\theta) \, d\theta$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \left(2\cos n\theta + \cos 3\theta \cdot \cos n\theta - i \sin 4\theta \cdot \sin n\theta \right) d\theta$$

$$= \frac{1}{\pi} \int_0^{\pi} \left\{ 2\cos n\theta + \frac{\cos (n+3)\theta + \cos (n-3)\theta}{2} - i \cdot \frac{\cos (n-4)\theta - \cos (n+4)\theta}{2} \right\} d\theta$$

OBS
$$\int_0^{\pi} \cos k\theta \, d\theta = \begin{cases} \pi, & k = 0 \\ 0, & \text{ANNARS} \end{cases}$$

$$\Rightarrow c_n = 0 \quad \text{om} \quad n \notin \{0, \pm 3, \pm 4\}$$

$$c_0 = \frac{2}{\pi} \cdot \pi = 2, \quad c_{\pm 3} = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}, \quad c_{\pm 4} = -i \cdot \frac{1}{\pi} \cdot \frac{\pm\pi}{2} = \mp \frac{i}{2}$$

$$\Rightarrow u(r,\theta) = 2 + \frac{1}{2} \cdot r^3 \cdot e^{i3\theta} + \frac{1}{2} \cdot r^{|-3|} \cdot e^{-i3\theta} - \frac{i}{2} \cdot r^4 \cdot e^{i4\theta} + \frac{i}{2} \cdot r^{|-4|} \cdot e^{-i4\theta}$$

$$= 2 + \frac{r^3}{2}\left(e^{i3\theta} + e^{-i3\theta}\right) + i \cdot \frac{r^4}{2}\left(-e^{i4\theta} + e^{-i4\theta}\right)$$

$$= 2 + r^3 \cdot \cos 3\theta + r^4 \cdot \sin 4\theta$$

SVAR: $u(r,\theta) = 2 + r^3 \cdot \cos 3\theta + r^4 \sin 4\theta$