

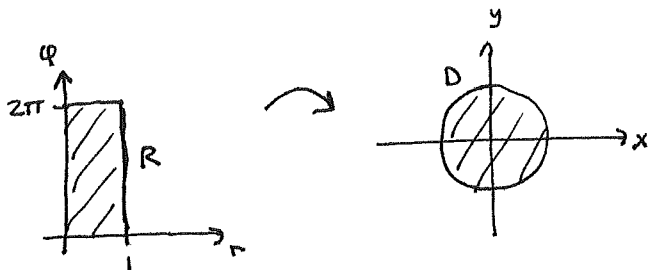
6.20

LÄS  $D = \{(x,y) : x^2 + y^2 < 1\}$ , BERÄKNA

$$I := \iint_D \frac{(x+y)^2}{1+x^2+y^2} dx dy = \iint_D f(x,y) dx dy$$

BYT TILL POLÄRA KOORDINATER

$$\begin{cases} x = r \cdot \cos \varphi =: g(r, \varphi) \\ y = r \cdot \sin \varphi =: h(r, \varphi) \end{cases}$$



ENLIGT SATS 6 (S. 261)

$$I = \iint_D f(x,y) dx dy = \iint_R f(g(r,\varphi), h(r,\varphi)) \cdot |J(r,\varphi)| \cdot dr d\varphi$$

DÄR

$$|J(r,\varphi)| = \left| \frac{d(x,y)}{d(r,\varphi)} \right| = \begin{vmatrix} \frac{\partial g}{\partial r} & \frac{\partial g}{\partial \varphi} \\ \frac{\partial h}{\partial r} & \frac{\partial h}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{vmatrix}$$

$$= |r \cdot \cos^2 \varphi + r \sin^2 \varphi| = |r (\cos^2 \varphi + \sin^2 \varphi)| = r$$

SÅLEDES

$$I = \int_{r=0}^1 \int_{\varphi=0}^{2\pi} \frac{\overbrace{(r \cos \varphi + r \sin \varphi)^2}^{= r^2 (\cos^2 \varphi + \sin^2 \varphi + 2 \cos \varphi \sin \varphi)}}{1+r^2} \cdot r dr d\varphi$$

$$= \int_{r=0}^1 \int_{\varphi=0}^{2\pi} \frac{r^3}{1+r^2} \cdot \underbrace{(1+2 \cos \varphi \sin \varphi)}_{= \sin(2\varphi)} dr d\varphi$$

$$= \int_{r=0}^1 \frac{r^3}{1+r^2} dr \cdot \int_{\varphi=0}^{2\pi} (1+\sin 2\varphi) d\varphi$$

$$= \int_0^{2\pi} d\varphi + \int_0^{2\pi} \sin 2\varphi d\varphi = 2\pi$$

$$= \int_0^1 \left( r - \frac{r}{1+r^2} \right) dr \cdot 2\pi = 2\pi \left[ \frac{r^2}{2} - \frac{1}{2} \ln(1+r^2) \right]_0^1 = 2\pi \cdot \left( \frac{1}{2} - \frac{1}{2} \ln 2 \right)$$

$$= \pi \cdot (1 - \ln 2)$$

POLYNOMDIVISION:

$$\frac{r}{1+r^2} = \frac{r^3}{1+r^2} \Rightarrow \frac{r^3}{1+r^2} = r - \frac{r}{1+r^2}$$

6.19 LÄT  $D = \{(x,y) : |x| + |y| \leq 1\}$ , BERÄKNA

$$I := \iint_D (x^2 - y^2)^{10} dx dy$$

BYT KOORDINATER

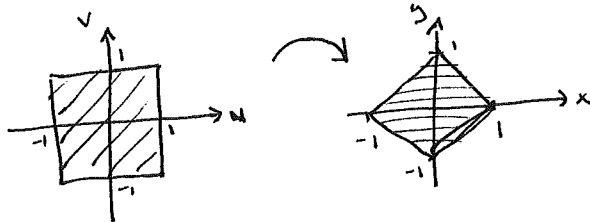
$$\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$$

$$\Leftrightarrow \begin{cases} x+y = u \\ x-y = v \end{cases}$$

$$\left| \frac{d(x,y)}{d(u,v)} \right| = \frac{1}{2} \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \frac{1}{2} \cdot \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$$

BESTÄM INTEGRATIONS-OMRÅDE GENOM ATT BETRÄKTA RANDEN AV  $D$ :

$$\begin{aligned} x+y &= 1 &\Rightarrow u &= 1 \\ -x-y &= 1 &\Rightarrow u &= -1 \\ x-y &= 1 &\Rightarrow v &= 1 \\ -x+y &= 1 &\Rightarrow v &= -1 \end{aligned}$$



VI FÖR

$$I = \int_{u=-1}^1 \int_{v=-1}^1 \left( \frac{1}{4}(u+v)^2 - \frac{1}{4}(u-v)^2 \right)^{10} \cdot \frac{1}{2} \cdot du dv = \frac{1}{2} \int_{u=-1}^1 \int_{v=-1}^1 (u \cdot v)^{10} du dv$$

$$= \frac{1}{2} \left( \left[ \frac{u^{11}}{11} \right]_{-1}^1 \right)^2 = \frac{1}{2} \left( \frac{1}{11} + \frac{1}{11} \right)^2 = \frac{2}{11^2}$$

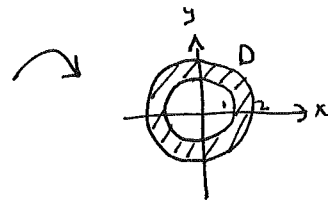
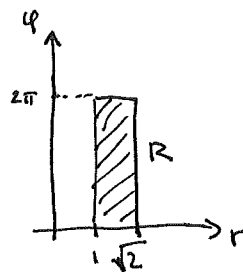
6.21 LÄT  $D = \{ (x,y) : 1 \leq x^2 + y^2 \leq 2 \}$ , BERÄKNA

$$I := \iint_D \ln(1+x^2+y^2) dx dy$$

BYT TILL POLÄRA KOORDINATER

$$\begin{cases} x = r \cdot \cos \varphi \\ y = r \cdot \sin \varphi \end{cases}$$

VI FÅR ATT



$$\begin{aligned} 1 \leq x^2 + y^2 \leq 2 \\ \Downarrow \\ 1 \leq r^2 \leq 2 \end{aligned}$$

$$I = \iint_D \ln(1+r^2) r dr d\varphi$$

$$= \int_{r=1}^{\sqrt{2}} r \cdot \ln(1+r^2) dr \cdot \int_{\varphi=0}^{2\pi} d\varphi = 2\pi \cdot \left\{ \left[ \frac{r^2}{2} \ln(1+r^2) \right]_1^{\sqrt{2}} - \frac{1}{2} \int_1^{\sqrt{2}} \frac{r^2 \cdot 2r}{1+r^2} dr \right\}$$

$$= 2\pi \cdot \left\{ \frac{1}{2} \cdot (2 \ln 3 - \ln 2) - \int_1^{\sqrt{2}} \frac{r^3}{1+r^2} dr \right\}$$

POLYNOM DIVISION GER

$$\frac{1+r^2 \overbrace{r}^r}{\underbrace{-(r+r^3)}_{-r}} \Rightarrow \frac{r^3}{1+r^2} = r - \frac{r}{1+r^2}$$

$$\int_1^{\sqrt{2}} \frac{r^3}{1+r^2} dr = \int_1^{\sqrt{2}} \left( r - \frac{r}{1+r^2} \right) dr = \left[ \frac{r^2}{2} - \frac{1}{2} \ln(1+r^2) \right]_1^{\sqrt{2}} = \frac{1}{2} \cdot (2 - \ln 3 - 1 + \ln 2)$$

VILLET GER

$$I = 2\pi \cdot \left\{ \frac{1}{2} (2 \ln 3 - \ln 2) - \frac{1}{2} (2 - \ln 3 - 1 + \ln 2) \right\}$$

$$= \frac{3}{2} \pi \cdot (3 \ln 3 - 2 \ln 2 - 1) = \pi \cdot \left( \ln \frac{3^3}{2^2} - 1 \right) = \pi \cdot \left( \ln \frac{27}{4} - 1 \right)$$

6.25

LÄT  $D = \{ (x,y) : x^2 + y^2 - 2x + 6y \leq 6 \}$ , BERÄKNA

$$I := \iint_D xy \, dx \, dy$$

KVADRATKOMPLETTERA:

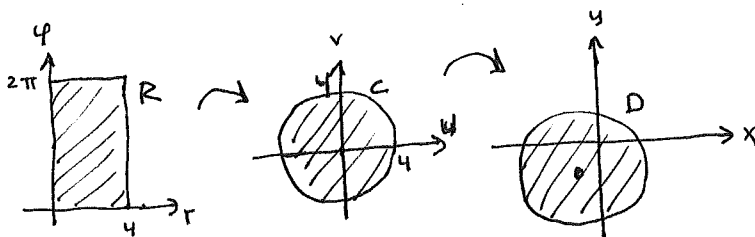
$$\begin{aligned} x^2 - 2x + y^2 + 6y - 6 &= (x-1)^2 - 1 + (y+3)^2 - 9 - 6 \\ &= (x-1)^2 + (y+3)^2 - 16 \end{aligned}$$

VI SER ATT  $D$  ÄR EN CIRKEL MED RADIE 4 OCH CENTRUM I PUNKTEN  $(1, -3)$ .

GÖR KOORDINATBYTTE I TVÅ STEG:

$$\begin{cases} x = u + 1 \\ y = v - 3 \end{cases}$$

$$\begin{cases} u = r \cdot \cos \varphi \\ v = r \cdot \sin \varphi \end{cases}$$



VI FÄR ATT

$$I = \iint_C (u+1)(v-3) \cdot \left| \frac{d(x,y)}{d(u,v)} \right| du \, dv = \iint_R (r \cos \varphi + 1)(r \sin \varphi - 3) \left| \frac{d(x,y)}{d(u,v)} \right| \cdot r \cdot dr \, d\varphi$$

$$= \int_{r=0}^4 \int_{\varphi=0}^{2\pi} (r \cos \varphi + 1)(r \sin \varphi - 3) \cdot r \cdot dr \, d\varphi$$

$$= \int_{r=0}^4 \int_{\varphi=0}^{2\pi} r \cdot (r^2 \cos \varphi \sin \varphi - 3r \cos \varphi + r \sin \varphi - 3) dr \, d\varphi$$

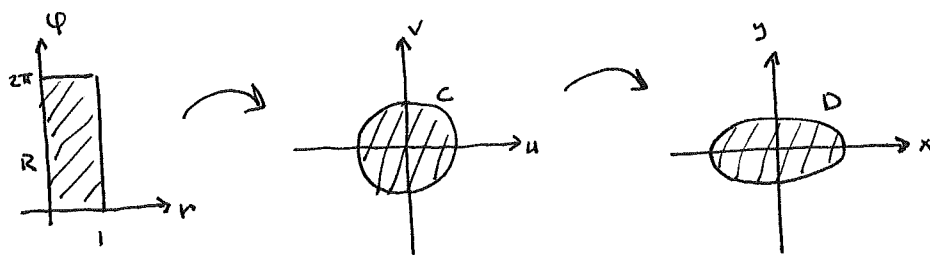
$$= \int_{r=0}^4 r \left( \underbrace{\frac{r^2}{2} \int_0^{2\pi} \sin 2\varphi \, d\varphi}_{=0} - 3r \underbrace{\int_0^{2\pi} \cos \varphi \, d\varphi}_{=0} + r \underbrace{\int_0^{2\pi} \sin \varphi \, d\varphi}_{=0} - 3 \cdot 2\pi \right) dr$$

$$= -6\pi \cdot \int_0^4 r \, dr = -6\pi \cdot \left[ \frac{r^2}{2} \right]_0^4 = -6\pi \cdot \frac{16}{2} = -48\pi$$

6.26

LÄT  $D = \{(x,y) : 2x^2 + 3y^2 \leq 1\}$ , BERÄKNA

$$I := \iint_D (x^2 + y^2) dx dy$$



BYT KOORDINATER I TVÅ STEG

$$\begin{cases} u = \sqrt{2} \cdot x \\ v = \sqrt{3} \cdot y \end{cases} \iff \begin{cases} x = \frac{u}{\sqrt{2}} \\ y = \frac{v}{\sqrt{3}} \end{cases} \quad \begin{matrix} \text{(AVBILDAR ENHETSSKIVAN)} \\ \text{PÅ } D \end{matrix}$$

$$\begin{cases} u = r \cdot \cos \varphi \\ v = r \cdot \sin \varphi \end{cases} \quad \begin{matrix} \text{(AVBILDAR REKTANGELN} \\ R \text{ PÅ ENHETSSKIVAN)} \end{matrix}$$

FUNKTIONAL DETERMINANTERNA GES AV

$$\left| \frac{\partial(u,v)}{\partial(r,\varphi)} \right| = r, \quad \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\sqrt{6}}$$

ENLIGT SATS 6 (S. 261) FÄR VI

$$\begin{aligned} I &= \iint_D \left( \frac{u^2}{2} + \frac{v^2}{3} \right) \cdot \frac{1}{\sqrt{6}} du dv = \iint_R \left( \frac{r^2 \cos^2 \varphi}{2} + \frac{r^2 \sin^2 \varphi}{3} \right) \cdot \frac{1}{\sqrt{6}} \cdot r dr d\varphi \\ &= \frac{1}{\sqrt{6}} \int_{\varphi=0}^{2\pi} \int_{r=0}^1 r^3 \left( \frac{1}{2} \cos^2 \varphi + \frac{1}{3} \sin^2 \varphi \right) dr d\varphi = \frac{1}{\sqrt{6}} \int_{r=0}^1 r^3 dr \cdot \left\{ \frac{1}{2} \int_0^{2\pi} \cos^2 \varphi d\varphi + \frac{1}{3} \int_0^{2\pi} \sin^2 \varphi d\varphi \right\} \\ &= \frac{1}{\sqrt{6}} \cdot \left[ \frac{r^4}{4} \right]_0^1 \cdot \left\{ \int_0^{2\pi} \left( \frac{1}{2} \cos^2 \varphi + \frac{1}{3} (1 - \cos^2 \varphi) \right) d\varphi \right\} \\ &= \frac{1}{4\sqrt{6}} \cdot \int_0^{2\pi} \left( \frac{1}{3} + \frac{1}{6} \cos^2 \varphi \right) d\varphi = \frac{1}{4\sqrt{6}} \cdot \left( \frac{2\pi}{3} + \frac{1}{6} \int_0^{2\pi} \cos^2 \varphi d\varphi \right) \\ &= \frac{1}{4\sqrt{6}} \cdot \left( \frac{2\pi}{3} + \frac{\pi}{6} \right) = \frac{5\pi}{4 \cdot 6\sqrt{6}} = \frac{5\pi\sqrt{6}}{144} \end{aligned}$$

OBS:  $\int_0^{2\pi} \cos^2 \varphi d\varphi = \left[ \sin \varphi \cdot \cos \varphi \right]_0^{2\pi} - \int_0^{2\pi} \sin \varphi \cdot (-\sin \varphi) d\varphi = \int_0^{2\pi} (1 - \cos^2 \varphi) d\varphi \implies 2 \int_0^{2\pi} \cos^2 \varphi d\varphi = 2\pi$

6.29 LÄT  $D = \{ (x,y) : 1 < x^2 - y^2 < 4, \sqrt{17} < x^2 + y^2 < 5, x < 0, y > 0 \}$ ,

BERÄKNA

$$I := \iint_D (x^4 - y^4) dx dy$$

INFÖR KOORDINATERNA

$$\begin{cases} u = x^2 - y^2 \\ v = x^2 + y^2 \end{cases} \quad \left| \frac{d(u,v)}{d(x,y)} \right| = \begin{vmatrix} 2x & -2y \\ 2x & 2y \end{vmatrix} = 4xy + 4xy = 8xy$$

(FUNKTIONAL DETERMINANTEN ÄR NOLLSKILD PÅ D SÅ DETTA ÄR ETT GILTIGT KOORDINATBYTE.)

BESTÄM NYTT INTEGRATIONSOMRÅDE

$$1 < x^2 - y^2 < 4 \iff 1 < u < 4$$

$$\sqrt{17} < x^2 + y^2 < 5 \iff \sqrt{17} < v < 5$$

$$\begin{cases} u+v = 2x^2 \\ v-u = 2y^2 \end{cases} \iff \begin{cases} x^2 = \frac{1}{2}(u+v) \\ y^2 = \frac{1}{2}(v-u) \end{cases}$$

$$\begin{cases} x < 0 \\ y > 0 \end{cases} \implies \begin{cases} u+v > 0 & (\text{OK}) \\ v-u > 0 & (\text{OK, } \sqrt{17} > 4) \end{cases}$$

D.V.S. VI VILL INTEGRERA ÖVER  $E = \{ (u,v) : 1 < u < 4, \sqrt{17} < v < 5 \}$

$$I = \iint_E \left( \frac{1}{4}(u+v)^2 - \frac{1}{4}(v-u)^2 \right) \cdot \left| \frac{d(x,y)}{d(u,v)} \right| =$$

$$= \frac{1}{4} \iint_E (u^2 + 2uv + v^2 - v^2 + 2uv - u^2) \cdot \frac{1}{8xy} du dv$$

$$= \frac{1}{8} \int_{u=1}^4 \int_{v=\sqrt{17}}^5 u \cdot v \cdot \left( \frac{1}{2} \sqrt{(u+v)(v-u)} \right)^{-1} du dv = \frac{1}{4} \int_{u=1}^4 \int_{v=\sqrt{17}}^5 \frac{uv}{\sqrt{v^2 - u^2}} du dv$$

$$= \frac{1}{4} \int_{v=\sqrt{17}}^5 v \left[ -\sqrt{v^2 - u^2} \right]_{u=1}^4 dv = \frac{1}{4} \int_{\sqrt{17}}^5 v \left( \sqrt{-16+v^2} + \sqrt{-1+v^2} \right) dv$$

$$= \frac{1}{4} \left( \frac{1}{3} \left[ (v^2-1)^{3/2} - (v^2-16)^{3/2} \right]_{\sqrt{17}}^5 \right) = \frac{1}{12} \left( 24^{3/2} - 9^{3/2} - 16^{3/2} + 1 \right)$$

$$= \frac{1}{12} \cdot \left( 6^{3/2} \cdot 2^3 - 27 - 64 + 1 \right) = \frac{6^{3/2} \cdot 8}{2 \cdot 6} - \frac{90}{12} = \frac{1}{2} \cdot \left( 8 \cdot 6^{1/2} - 15 \right) = \frac{1}{2} (8\sqrt{6} - 15)$$

## SAMMANFATTNING — LEKTION 8

### BEGREPP:

- VARIABELBYTE (s. 259)
- FUNKTIONALDETERMINANT (s. 140)  
(TOLKNING SOM AREA FÖRSTÖRING s. 260)

### SATSER:

- VARIABELBYTE I DUBBELINTEGRAL, SATS 6 (s. 261)
- FUNKTIONALDETERMINANT FÖR SAMMANSÄTTNING,  
SAT S 1 (s. 141),
- INVERSENS FUNKTIONALDETERMINANT, EKV. (20) (s. 141)