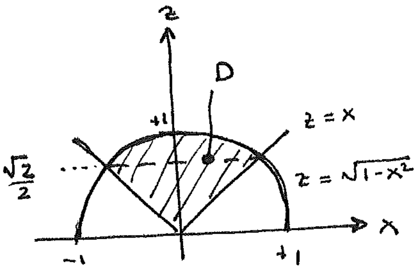
LEKTION 10 22/2/2008 SF1646

BJÖRN WINCKLER
winckler@kth.se

$\boxed{7.4}$ LÅT $D = \{ (x,y,z) : \ x^2+y^2+z^2 \leq 1, \quad z \geq \sqrt{x^2+y^2} \}$, BERÄKNA

$$I := \iiint_D \frac{z \, dx\,dy\,dz}{1+x^2+y^2}$$



VI DELAR UPP BERÄKNINGEN I TVÅ DELAR

$$I = I_1 + I_2$$

DÄR

$$I_1 = \int_{z=0}^{\sqrt{2}/2} \left(\iint_{x^2+y^2 \leq z^2} \frac{z}{1+x^2+y^2} \, dx\,dy \right) dz$$

$$= \int_{z=0}^{\sqrt{2}/2} z \cdot \left(\int_{\varphi=0}^{2\pi} \int_{r=0}^{z} \frac{r \cdot dr}{1+r^2} \, d\varphi \right) dz$$

(BYTE TILL POLÄRA KOORDINATER I X-Y-PLANET)

$$= \int_{z=0}^{\sqrt{2}/2} z \cdot 2\pi \cdot \frac{1}{2} \Big[\log(1+r^2) \Big]_{r=0}^{z} dz$$

$$= \pi \cdot \int_0^{\sqrt{2}/2} z \cdot \log(1+z^2) \, dz = \frac{\pi}{2} \cdot \Big[(1+z^2)\log(1+z^2) - z^2 \Big]_0^{\sqrt{2}/2}$$

$$= \frac{\pi}{2} \left\{ (1+1/2) \cdot \log(1+1/2) - 1/2 \right\} = \frac{3\pi}{4} \cdot \log(3/2) - \frac{\pi}{4}$$

OCH

$$I_2 = \int_{z=\sqrt{2}/2}^{1} \left(\iint_{x^2+y^2 \leq 1-z^2} \frac{z \, dx\,dy}{1+x^2+y^2} \right) dz$$

$$= \int_{z=\sqrt{2}/2}^{1} z \cdot \left(\int_{\varphi=0}^{2\pi} \int_{r=0}^{\sqrt{1-z^2}} \frac{r\,dr}{1+r^2} \, d\varphi \right) dz = \ldots$$

(FORTS. NÄSTA SIDA)

$\boxed{7.4}$

(FORTS.)

$$I_2 = \int_{z=\sqrt{2}/2}^{1} z \cdot 2\pi \cdot \frac{1}{2} \left[\log\left(1+r^2\right) \right]_{r=0}^{\sqrt{1-z^2}} dz$$

$$= \pi \cdot \int_{\sqrt{2}/2}^{1} z \cdot \log\left(2-z^2\right) dz = -\frac{\pi}{2} \left[\left(2-z^2\right) \cdot \log\left(2-z^2\right) + z^2 \right]_{\sqrt{2}/2}^{1}$$

$$= -\frac{\pi}{2}\left(1 - \frac{3}{2} \cdot \log\frac{3}{2} - \frac{1}{2}\right)$$

SÅLEDES FÅR VI ATT

$$I = I_1 + I_2 =$$

$$= \frac{3\pi}{4} \log\frac{3}{2} - \frac{\pi}{4} + \frac{3\pi}{4} \cdot \log\frac{3}{2} - \frac{\pi}{4} =$$

$$= \frac{3\pi}{2} \cdot \log\frac{3}{2} - \frac{\pi}{2} = \frac{\pi}{2} \cdot \left(3 \cdot \log\frac{3}{2} - 1\right)$$

$\boxed{7.13}$ LÅT $D = \{ (x,y,z) : x^2+y^2+z^2 \le 1, \ z \ge \sqrt{x^2+y^2} \}$, BERÄKNA

$$I := \iiint_D \sqrt{x^2+y^2+z^2} \, dx\,dy\,dz$$

BYT TILL SFÄRISKA KOORDINATER

$$\begin{cases} x = r \cdot \sin\theta \cdot \cos\varphi \\ y = r \cdot \sin\theta \cdot \sin\varphi \\ z = r \cdot \cos\theta \end{cases} \qquad \left| \frac{d(x,y,z)}{d(r,\theta,\varphi)} \right| = r^2 \cdot \sin\theta \qquad (\text{EX } 13, \text{ s.} 140)$$

DET NYA INTEGRATIONSOMRÅDET GES AV

$$x^2+y^2+z^2 \le 1 \iff r^2 \le 1$$

$$z \ge \sqrt{x^2+y^2} \iff 0 \le \theta \le \frac{\pi}{4}$$

SÅLEDES

$$I = \int_{\theta=0}^{\pi/4} \int_{\varphi=0}^{2\pi} \int_{r=0}^{1} r \cdot r^2 \cdot \sin\theta \, dr\,d\theta\,d\varphi$$

$$= \int_{\theta=0}^{\pi/4} \sin\theta \, d\theta \cdot \int_{\varphi=0}^{2\pi} d\varphi \cdot \int_{r=0}^{1} r^3 \, dr = \Big[-\cos\theta \Big]_0^{\pi/4} \cdot 2\pi \cdot \Big[\frac{r^4}{4} \Big]_0^1$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right) \cdot 2\pi \cdot \frac{1}{4} = \frac{(2-\sqrt{2})\pi}{4}$$

SAMMANFATTNING – LEKTION 10

BEGREPP

- TRIPPELINTEGRALER (s. 285), TOLKNING:

$$\iiint_D dx\,dy\,dz = \text{Volym}(D)$$ (s. 286) (INTEGRATION AV FUNKTIONEN $f=1$)

- RYMDPOLÄRA KOORDINATER (SFÄRISKA KOORDINATER), EKV. (5) (s. 291)

SATSER:

- BERÄKNING AV TRIPPELINTEGRAL GENOM ITERERAD INTEGRERING, EKV. (1) & (2) (s. 287)

- BERÄKNING AV TRIPPELINTEGRAL GENOM VARIABELSUBSTITUTION, EKV. (3) (s. 288)