Effective evaluation of optimal stability polynomials

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Runge-Kutta methods are used for numerical integration of ordinary differential equations for more than a hundred years already. The construction of n-steps stable explicit RK-method of the p-th degree of accuracy gives rise to the following optimization problem which is at least 40 years old.

Among all degree $\leq n$ polynomials that approximate the exponential function at $x = 0$ to the given order $p \leq n$, e.g. $P_n(x) = 1 + x + x^2/2 + ... + x^p/p! + o(x^p)$, find the one that remains in the limits $-1 \leq P_n(x) \leq 1$ on the maximal interval $[-L, 0]$, $L > 0$.

When $p = 1$, the solution may be expressed in terms of classical Chebyshev polynomial $P_n(x) = T_n(1 + x/n^2)$. When $p = 2$ the optimal stability polynomial $P_n(x)$ is the more involved Zolotarev polynomial that may be evaluated in terms of elliptic functions. Many authors note that in case $p > 2$ no analytical formula for the solution is known. Several methods of numerical optimization for this problem were proposed. Unfortunately they all are very labour- intensive and become inefficient when the degree n of the polynomial is equal to several tens.

We suggest an explicit analytical formula for the optimal stability polynomials. It is numerically effective. Moreover, the amount of calculations does not rise with the growth of n at all. Our approach is based on the theory of Riemann surfaces and the so-called g-extremal polynomials, generalizations of classical Chebyshev and Zolotarev polynomials.

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