We consider the Fourier multipliers acting from $H^p$ to $H^q$, $0 < p \leq q \leq 1$ in most general case of tube domains over open cones:

$$T_\Gamma = \{z \in \mathbb{C}^n, z = x + iy : x \in \mathbb{R}^n, y \in \Gamma\}.$$

Here $\Gamma$ is an open cone in $\mathbb{R}^n$.

If the set $\Gamma^* = \{x \in \mathbb{R}^n : (x,t) \geq 0 \,\forall t \in \Gamma\}$ has nonempty interior then it is a closed cone that is called a *conjugate cone* to $\Gamma$. In this case the cone $\Gamma$ is said to be an *acute open cone.*

The *Fourier transform* of a function $f \in H^p(T_\Gamma)$, $p \in (0,1]$, is defined by

$$\widehat{f}(\xi) = e^{2\pi(\xi,\delta)} \widehat{f(\cdot + i\delta)}(\xi), \quad \xi \in \mathbb{R}^n,$$

where $\delta \in \Gamma$ is chosen arbitrarily (see [2]).

The following inversion formula holds true

$$f(z) = \int_{\Gamma^*} \widehat{f}(t)\, e^{2\pi i(z,t)}\, dt, \quad z \in T_\Gamma. \tag{1}$$

Let $\Gamma$ be an acute open cone in $\mathbb{R}^n$, $n \in \mathbb{N}$. A measurable function $\varphi : \Gamma^* \to \mathbb{C}$ is called a *multiplier* from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$, $0 < p \leq q \leq 1$ (designation: $\varphi \in M_{p,q}(T_\Gamma)$) if for any function $f \in H^p(T_\Gamma)$ the function $\varphi \cdot \widehat{f}$ coincides almost everywhere on $\Gamma^*$ with Fourier transform of some function $F_\varphi[f] \in H^q(T_\Gamma)$ and

$$\|\varphi\|_{M_{p,q}(T_\Gamma)} := \sup_{\|f\|_{H^p} \neq 0} \frac{\|F_\varphi[f]\|_{H^q}}{\|f\|_{H^p}} < \infty.$$

Let us note that, according to (1), the function $F_\varphi[f]$ in this definition is defined uniquely:

$$F_\varphi[f](z) = \int_{\Gamma^*} \varphi(t)\, \widehat{f}(t)\, e^{2\pi i(z,t)}\, dt, \quad z \in T_\Gamma.$$

Some properties of multipliers are established. Several useful conditions for multipliers are obtained. Let us cite the following theorem.

**Theorem 1** *Let $\Gamma$ be an acute open cone in $\mathbb{R}^n$; $\varphi \in C(\mathbb{R}^n)$ and there exists $\sigma > 0$ such that $\operatorname{supp} \varphi \subset [-\sigma, \sigma]^n$. If furthermore $\widehat{\varphi} \in L^q(\mathbb{R}^n)$ for some $q \in (0,1]$, then $\varphi \in M_{p,q}(T_\Gamma)$ for every $p \in (0,q]$ and*

$$\|\varphi\|_{M_{p,q}(T_\Gamma)} \leq \frac{\gamma(n,p,q)}{(V_n(\Gamma))^{1/p-1}} \sigma^{n(1/p-1)} \|\widehat{\varphi}\|_q.$$

*Here $\gamma$ denotes some positive constant depending only on parameters in parentheses, and $V_n(\Gamma)$ is the maximal volume of a simplex that can be constructed on $n$ unit vectors, contained in $\overline{\Gamma}$.*

In particular, any compactly supported function of the class $C^\infty(\mathbb{R}^n)$ belongs to $M_{p,q}(T_\Gamma)$ for any $p$ and $q$ such that $0 < p \leq q \leq 1$.

It is established that the operator of Bochner-Riesz type means

$$R_h^{r,\alpha}(f;z) = \int_{|x| \leq 1/h} \widehat{f}(x) \left(1 - h^{2r} |x|^{2r}\right)^\alpha e^{2\pi i(z,x)}\, dx, \quad z \in T_\Gamma,\ h > 0,$$

is bounded linear operator acting from $H^p(T_\Gamma)$ to $H^q(T_\Gamma)$, $0 < p \leq q \leq 1$, if and only if $\alpha > \frac{n}{q} - \frac{n+1}{2}$.

Let $K$ be a symmetric body in $\mathbb{R}^n$, $K^*$ denotes its polar, and $\mathcal{E}(K)$ denotes the class of all entire functions of exponential type $K$ (see [1, Ch. III, § 4]).

An analog of the classical Bernstein inequality for the class $\mathcal{E}(K^*) \cap H^p(T_\Gamma)$, $p \in (0,1]$, is obtained.

It is easily to see that Hardy spaces $H^p(T_\Gamma)$ and $H^q(T_\Gamma)$ are not enclosed one into another for different $p$ and $q$. However, if additionally we require the functions belong to $\mathcal{E}(K^*)$, then we have the following different metrics inequality.

**Theorem 2** *Let $\Gamma$ be an acute open cone in $\mathbb{R}^n$, $n \in \mathbb{N}$ and $K$ be a symmetric body in $\mathbb{R}^n$. If a function $f$ belongs to the class $\mathcal{E}(K^*) \cap H^p(T_\Gamma)$ for some $p \in (0,\infty)$ then*

$$\|f\|_{H^q} \leq \left(\frac{p}{2}+1\right)^{n(1/p-1/q)} \cdot (\mathrm{mes}\ (K \cap \Gamma^*))^{1/p-1/q} \cdot \|f\|_{H^p} \quad \forall q \in (p,\infty]$$

*(here* mes $A$ *is the Lebesgue measure of a set $A$). In particular, the class $\mathcal{E}(K^*) \cap H^p(T_\Gamma)$, $p \in (0,\infty)$, contains nonzero functions if and only if* mes $(K \cap \Gamma^*) > 0$.

# References


[1] Stein, E. M., Weiss, G. Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, NJ, 1971.

[2] Tovstolis, A. V. Fourier multipliers in Hardy spaces in tube domains over open cones and their applications, Methods Funct. Anal. Topology, **4**, No. 1 (1998), 68–89.