Some Developments and Applications of Local Time-Space Calculus

Raouf Ghomrasni*

Abstract

We show that the following limit $\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon}\int_0^t \big\{F(s,B_s-\varepsilon) \,-\, F(s,B_s+\varepsilon)\big\}\,ds$ is well defined for a large class of functions $F(t,\,x)$ and moreover we connect it with the integration with respect to local time L_t^x.

We give an illustrative example of the no continuity of the integration with respect to local time in the random case.

Key words and phrases: Itô's formula, local time, Brownian motion.
MSC2000: 60H05, 60J65.

1 Introduction

1.1. The local time of the Brownian motion B at the point a is defined as follows:

$$L_t^a = \mathbb{P}-\lim_{\varepsilon\downarrow 0} \frac{1}{2\,\varepsilon}\int_0^t 1_{(|B_s-a|\le\varepsilon)}\,ds$$

which equivalently could be written as follows:

$$L_t^a = \mathbb{P}-\lim_{\varepsilon\downarrow 0} \frac{1}{2\,\varepsilon}\int_0^t \big(1_{(B_s-\varepsilon\le a)} \,-\, 1_{(B_s+\varepsilon\le a)}\big)\,ds\,.$$

Here we are, more generally, interested in the limit in L^1

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\,\varepsilon}\int_0^t \big\{F(s,B_s-\varepsilon) \,-\, F(s,B_s+\varepsilon)\big\}\,ds$$

for some function F.
Our motivation come from the desire to connect Chitashvili and Mania results ([3]) with those of Eisenbaum ([5]).

1.2. We give an example which illustrates that the integration with respect to $(L_t^x; 0\le t\le 1, x\in \mathbb{R})$ does not admit a linear extension in the random case (see section 3.2 for details) and in particular local time is not a 1-integrator which is also proved by Eisenbaum ([5]).

1.3. The power of the local time-space calculus is illustrated by results concerning extensions of the Itô-Tanaka formula and a change-of-variable formula with local time on curves.

*Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000 Aarhus C, Denmark and Centre for Mathematical Physics and Stochastics (funded by the Danish National Research Foundation), e-mail: raouf@imf.au.dk

2 Notation and preliminaries

Let $B=(B_t)_{t\geq 0}$ be a standard Brownian motion and $(L_t^x; t\geq 0, x\in I\!R)$ be a continuous version of its local time process. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the natural filtration generated by B. Without loss of generality, we restrict our attention to functions defined on $[0,\,1]\times I\!R$.

For a measurable function f from $[0,\,1]\times I\!R$ into $I\!R$ define the norm $\|\,.\,\|$ by

$$\| f \| = 2\Big(\int_0^1\int_{I\!R} f^2(s,\,x)\,e^{-x^2/2s}\frac{ds\,dx}{\sqrt{2\pi s}}\Big)^{1/2} + \int_0^1\int_{I\!R}|xf(s,\,x)|e^{-x^2/2s}\,\frac{ds\,dx}{s\sqrt{2\pi s}}.$$

Let $\mathcal{H}$ be the set of functions f such that $\| f \| < \infty$.

In Eisenbaum [5], it is shown that the integration with respect to L is possible in the following sense. Let f_Δ be an elementary function on $[0,\,1]\times I\!R$, meaning that

$$f_\Delta(t,\,x) = \sum_{(s_i,x_j)\in\Delta} f_{i,j}1_{(s_i,s_{i+1}]}(t)\,1_{(x_j,x_{j+1}]}(x),$$

where $\Delta=\{(s_i,\,x_j),\,1\leq i\leq n,\,1\leq j\leq m\}$ is an $[0,\,1]\times I\!R$ grid, and, for every $(i,\,j)$, f_{ij} is in $I\!R$. For such a function, integration with respect to L is defined by

$$\int_0^1\int_{I\!R} f_\Delta(s,\,x)\,dL_s^x \;=\; \sum_{(s_i,x_j)\in\Delta} f_{i,j}(L_{s_{i+1}}^{x_{j+1}} - L_{s_i}^{x_{j+1}} - L_{s_{i+1}}^{x_j} + L_{s_i}^{x_j})\,.$$

Let f be an element of $\mathcal{H}$. For any sequence of elementary functions $(f_{\Delta_k})_{k\in I\!N}$ converging to f in $\mathcal{H}$, the sequence $(\int_0^1\int_{I\!R} f_{\Delta_k}(s,\,x)\,dL_s^x)_{k\in I\!N}$ converges in L^1. The limit obtained does not depend of the choice of the sequence (f_{Δ_k}) and represents the integral $\int_0^1\int_{I\!R} f(s,\,x)\,dL_s^x$.

Theorem 2.1 ([5]) *Let A be a random process such that for each x, $A(\cdot,x)$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and a.s. $\partial A/\partial t$ and $\partial A/\partial x$ exist and are continuous. Moreover a.s. $\partial A/\partial x$ is an element of $\mathcal{H}$, with bounded variations on compacts.*

Then for $t\geq 0$ we have

$$A(t,B_t) = A(0,B_0) + \int_0^t \frac{\partial A}{\partial t}(s,B_s)\,ds + \int_0^t \frac{\partial A}{\partial x}(s,B_s)\,dB_s - \frac{1}{2}\int_0^t\int_{I\!R}\frac{\partial A}{\partial x}(s,x)\,dL_s^x\,.$$

3 Main results

3.1 Deterministic case

Theorem 3.1 *Let F be a bounded element of $\mathcal{H}$. The following equalities hold in L^1:*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\Big\{F(s,B_s) - F(s,B_s-\varepsilon)\Big\}\,ds = -\int_0^t\int_{I\!R} F(s,x)\,d\,L_s^x \qquad (3.1)$$

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int_0^t\Big\{F(s,B_s+\varepsilon) - F(s,B_s)\Big\}\,ds = -\int_0^t\int_{I\!R} F(s,x)\,d\,L_s^x \qquad (3.2)$$

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\,\varepsilon}\int_0^t\Big\{F(s,B_s-\varepsilon) - F(s,B_s+\varepsilon)\Big\}\,ds = \int_0^t\int_{I\!R} F(s,x)\,d\,L_s^x\,. \qquad (3.3)$$

Remark 3.1 1. If we take $F(t,x) = 1_{(x\le a)}$ in (6.3.1) we have the very definition of L_t^a
2. Eisenbaum [5] has shown that, for any borelian function $b(t)$:

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\,\varepsilon} \int_0^t 1_{(|B_s\,-\,b(s)|\,<\,\varepsilon)}\, ds = \int_0^t \int_{I\!R} 1_{(-\infty,\,b(s))}(x)\, d\,L_s^x \quad in \quad L^1$$

which corresponds to (6.3.3) with $F(t,x) = 1_{(x\le b(t))}$.

Proof: Define $H_\varepsilon(t,\,x) = \frac{1}{\varepsilon}\int_{x-\varepsilon}^{x} F(t,\,y)\,d\,y$. Then $H_\varepsilon \to F$ in $\mathcal{H}$ as $\varepsilon \downarrow 0$. On the one hand $\frac{\partial}{\partial\,x}H_\varepsilon(t,\,x) = \frac{1}{\varepsilon}\big\{F(t,x) - F(t,\,x-\varepsilon)\big\}$. It follows that (see Eisenbaum [5] Theorem 5.1 (ii)) $\int_0^t \int_{I\!R} H_\varepsilon(s,\,x)\,d\,L_s^x = -\frac{1}{\varepsilon}\int_0^t \big\{F(s,\,B_s) - F(s,\,B_s-\varepsilon)\big\}\,ds$. On the other hand $\int_0^t \int_{I\!R} H_\varepsilon(s,\,x)\,d\,L_s^x \longrightarrow \int_0^t \int_{I\!R} F(s,\,x)\,d\,L_s^x$ in L^1.

□

Corollary 3.1 ([12]) *The following relation holds in* L^1*:*

$$\lim_{\varepsilon\downarrow 0} \frac{1}{2\,\varepsilon} \int_0^t g(s)\,I(b(s)-\varepsilon <\, B_s\, < b(s)+\varepsilon)\,ds = \int_0^t g(s) d\,L_s^b$$

for a continuous function $g:\,[0,\,t] \to I\!R$ *and a continuous curve* $b(\cdot)$ *with bounded variation on* $[0,\,t]$.

Proof: We apply Theorem 6.3.1 to the function $F(t,x) = g(t)\,I(x\,<\,b(t))$. It follows: $\frac{1}{2\,\varepsilon}\int_0^t g(s)\,I(b(s)-\varepsilon <\, B_s\, < b(s)+\varepsilon)\,ds \to \int_0^t\int_{I\!R} g(s)\,I(x\,<\,b(s))\,d\,L_s^x$ in L^1 as $\varepsilon \downarrow 0$. We conclude using Corollary 2.9 ([13]) that for the *continuous* function g we have $\int_0^t\, g(s)\,d\,L_s^{b(s)} = \int_0^t\, g(s)\,d\,L_s^b$. □

3.2 Random function case

Let a, b in $I\!R$ with $a < b$. Let $\mathcal{M}$ be the set of elementary processes A such that

$$A(s,\,x) \;=\; \sum_{(s_i,x_j)\in\Delta} A_{ij}\, 1_{(s_i,s_{i+1}]}(s)\, 1_{(x_j,x_{j+1}]}(x)\,,$$

where $(s_i)_{1\le i\le n}$ is a subdivision of $(0,\,1]$, $(x_j)_{1\le j\le m}$ is a finite sequence of real numbers in $(a,\,b]$, $\Delta = \{(s_i,\,x_j),\, 1\le i\le n,\, 1\le j\le m\}$, and A_{ij} an $\mathcal{F}_{s_j}$-measurable random variable such that $|A_{ij}| \le 1$ for every $(i,\,j)$.
Eisenbaum [5] asked the following question: Does integration with respect to $(L_t^x;\, 0\le t\le 1,\, x\in I\!R)$ admit a *linear* extension to $\mathcal{P}$ the field generated by $\mathcal{M}$, verifying the following property:

If $(A_n)_{n\ge 0}$ converges a.e. to $A(t,\,x)$, then $(\int_0^1 \int_a^b A_n(s,\,x)\,d\,L_s^x)_{n\ge 0}$ converges in L^1 to $\int_0^1 \int_a^b A(s,x)dL_s^x$.
She only obtained a negative answer to the following *weaker* question:

$$\text{Is the set} \left\{ \int_0^1 \int_a^b A(s,x)\,dL_s^x,\; A\in\mathcal{M} \right\} \text{bounded in } L^1\,?$$

Consequently integration with respect to $(L_t^x;\, 0 \le t \le 1,\, x \in I\!R)$ does not admit a *continuous* extension in L^1.
Here we give an *illustrative example*, thanks to a result obtained by Walsh, which shows the lack of a *linear* extension.

Let us define $A_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{x-\varepsilon}^{x} L_t^y\,dy$ and $\tilde{A}_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{x}^{x+\varepsilon} L_t^y\,dy$. We see easily that $A_\varepsilon(t,x)$ (resp. $\tilde{A}_\varepsilon(t,x)$) converges a.e. to L_t^x, nevertheless we have:

$$\lim_{\varepsilon\downarrow 0}\int_0^t\int_{I\!R} A_\varepsilon(s,x)\,d\,L_s^x \neq \lim_{\varepsilon\downarrow 0}\int_0^t\int_{I\!R} \tilde{A}_\varepsilon(s,x)\,d\,L_s^x\,.$$

Let us recall, for the convenience of the reader, Walsh's theorem about the decomposition of $A(t,B_t) := \int_0^t 1_{\{B_s\le B_t\}}\,ds$.

Theorem 3.2 ([14]) *$A(t,B_t)$ has the decomposition*

$$A(t,B_t) \;=\; \int_0^t L_s^{B_s}\,dB_s \;+\; X_t$$

where

$$\begin{aligned} X_t &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\,\varepsilon}\int_0^t \left\{L_s^{B_s} - L_s^{B_s-\varepsilon}\right\}ds \\ &= t + \lim_{\varepsilon\downarrow 0}\frac{1}{2\,\varepsilon}\int_0^t \left\{L_s^{B_s+\varepsilon} - L_s^{B_s}\right\}ds \end{aligned}$$

The limits exist in probability, uniformly for t in compact sets.

Our example follows by recalling the following property:

$$\int_0^t\int_{I\!R} A_\varepsilon(s,x)\,d\,L_s^x \;=\; -\frac{1}{\varepsilon}\int_0^t \left\{L_s^{B_s} - L_s^{B_s-\varepsilon}\right\}ds\,.$$

Proposition 3.1 (A more explicit decomposition of $A(t,B_t)$) *Let $B = (B_t)_{t\ge 0}$ be a standard Brownian motion and $(L_t^x; x \in I\!R, t \ge 0)$ a continuous version of its local time process. Let $A(t,x) = \int_{-\infty}^x L_t^y\,dy$.*
Then $A(t,B_t)$ is a Dirichlet process and has the following decomposition:

$$A(t,B_t) = t + \int_0^t L_s^{B_s}\,dB_s - \frac{1}{2}\int_0^t\int_{I\!R} L_s^x\,dL_s^x\,.$$

Moreover we have:

$$-\frac{1}{2}\int_0^t\int_{I\!R} L_s^x\,dL_s^x = I\!P\!-\!\lim_{\varepsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t \{L_s^{B_s+\epsilon} - L_s^{B_s}\}\,ds\,.$$

Proof: As noted by J. Walsh [14] $A(t,x)$ is continuously differentiable in t as long as $B_t \neq x$. Indeed, we have $A(t,x) = \int_0^t 1_{\{B_t\le x\}}\,ds$ so $A_t(t,x) = 1_{(B_t\le x)}$. Moreover, $A(t,x)$ is continuously differentiable in x and $A_x(t,x) = L_t^x$, but the second derivative fails to exist and this thanks to Eisenbaum's theorem (theorem 5.3 in [5]) doesn't matter.

□

3.3 Link with principal value

Theorem 3.3 *Let F be an absolutely continuous function on $\mathbb{R}$ such that F_x is absolutely continuous on $\mathbb{R} \setminus \{0\}$. Suppose that*

(i) there exists a limit $\alpha = \lim_{\varepsilon \downarrow 0} \big(F_x(\varepsilon) - F_x(-\varepsilon)\big)$*;*

(ii) $F_x \in L^2_{loc}(\mathbb{R})$

(iii) $x(F_x(x))^2 \longrightarrow 0$ *as* $x \to 0$.

Then a.s.

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\,\varepsilon} \int_0^t \big\{ F_x(B_s + \varepsilon) \,-\, F_x(B_s - \varepsilon) \big\}\, ds \;=\; \alpha\, L_t^0 \;+ \text{v.p.} \int_0^t F_{xx}(B_s)\, ds\,.$$

Proof: We set

$$\frac{1}{2\,\varepsilon} \int_0^t \big\{ F_x(B_s + \varepsilon) \,-\, F_x(B_s - \varepsilon) \big\}\, ds = I_\varepsilon + J_\varepsilon$$

where

$$I_\varepsilon := \frac{1}{2\,\varepsilon} \int_0^t 1_{(|B_s| > \varepsilon)} \big\{ F_x(B_s + \varepsilon) \,-\, F_x(B_s - \varepsilon) \big\}\, ds$$

$$J_\varepsilon := \frac{1}{2\,\varepsilon} \int_0^t 1_{(|B_s| \le \varepsilon)} \big\{ F_x(B_s + \varepsilon) \,-\, F_x(B_s - \varepsilon) \big\}\, ds\,.$$

Using (ii), (iii) and the fact that F_x is absolutely continuous in $\mathbb{R} \setminus \{0\}$, it follows:

$$\lim_{\varepsilon \downarrow 0} I_\varepsilon \;=\; \text{v.p.} \int_0^t F_{xx}(B_s)\, ds\,.$$

On the other hand:

$$\lim_{\varepsilon \downarrow 0} J_\varepsilon \;=\; \Big\{ \lim_{\varepsilon \downarrow 0} \big(F_x(\varepsilon) - F_x(-\varepsilon)\big) \Big\}\; L_t^0\,.$$

□

4 Applications

In the following two theorems we use Eisenbaum's formula to give a clean proof of two theorems originated from Nasyrov ([10]; theorem 9) and ([10]; theorem 10).

Theorem 4.1 (the generalized Tanaka formula)
For $t \ge 0$, $z > 0$, $y \in \mathbb{R}$, we have:

$$\frac{1}{2}(z \wedge L_t^y) \;=\; 1_{(\,L_t^y \le z)}\, (B_t - y)^+ \;-\; (B_0 - y)^+ - \int_0^t 1_{(B_s \ge y,\, L_s^y \le z)}\, dB_s\,.$$

Proof: We take the following function $F(x,t) = 1_{[0,\,z]}(t)\,(x-y)^+$. It follows that $F_x(x,t) = 1_{(t\le z)}\,1_{(x\ge y)}$. Hence

$$\int_0^t \int_{I\!R} F_x\,(x, L_s^y)\,d\,L_s^x \;=\; \int_0^t \int_{I\!R} 1_{(x\ge y)}\,1_{L_s^y\le z}\,dL_s^x \;=\; -\int_0^t 1_{(L_s^y\le z)}\,d_s\,L_s^y \;=\; -\,(z\wedge L_t^y)\,.$$

□

Theorem 4.2 (the generalized Skorokhod equation) *Let Φ a C^1 function. For $t\ge 0$, we have:*

$$\Phi(L_t^0)\,|B_t| \;=\; \Phi(0)\,|B_0| \;+\; \int_0^t sign(B_s)\,\Phi(L_s^0)\,dB_s + \int_0^{L_t^0} \Phi(z)\,dz\,.$$

In this case,

$$\int_0^{L_t^0} \Phi(z)\,dz \;=\; -\min_{0\le s\le t}\;min\Big(\int_0^s sign(B_u)\,\Phi(L_u^0)\,dB_u,\,0\Big)\,.$$

Proof: We have

$$-\frac{1}{2}\int_0^t \int_{I\!R} \Phi(L_s^0)\,sign(x)\,d\,L_s^x = -\frac{1}{2}\int_0^t \int_0^{\infty} \Phi(L_s^0)\,d\,L_s^x \;+\; \frac{1}{2}\int_0^t \int_{-\infty}^0 \Phi(L_s^0)\,d\,L_s^x$$

hence we obtain $-\frac{1}{2}\int_0^t \Phi(L_s^0)\,sign(x)\,d\,L_s^x \;=\; \int_0^t \Phi(L_s^o)\,d_s\,L_s^0\,.$

□

Theorem 4.3 ([11])

Let $B=(B_t)_{t\ge 0}$ be a standard Brownian motion and let $b: I\!R_+ \to I\!R$ be a continuous function of bounded variation. Setting $C = \{\,(s,x)\in I\!R_+\times I\!R \,|\, x<b(s)\,\}$ and $D = \{\,(s,x)\in I\!R_+\times I\!R \,|\, x>b(s)\,\}$ suppose that a continuous function $F: I\!R_+\times I\!R \to I\!R$ is given such that F is $C^{1,2}$ on $\overline{C}$ and F is $C^{1,2}$ on $\overline{D}$. Then we have:

$$\begin{aligned} F(t,B_t) = F(0,B_0) &+ \int_0^t F_t(s,B_s)\,ds + \int_0^t F_x(s,B_s)\,dB_s \\ &+ \frac{1}{2}\int_0^t F_{xx}(s,B_s)\,I(B_s\,\ne\,b(s))\,ds \\ &+ \frac{1}{2}\int_0^t \Big(F_x(s,b_s+)\;-\;F_x(s,b_s-)\Big)\,I(B_s\;=\;b(s))\,dL_s^b\,. \end{aligned}$$

Proof: We focus on the last term:

$$\int_0^t \int_{I\!R} F_x(s,\,x)\,dL_s^x \;=\; \int_0^t \int_{I\!R} F_x(s,\,x)\,1_{\{x\ne b(s)\}}\,dL_s^x \;+\; \int_0^t \int_{I\!R} F_x(s,\,x)\,1_{\{x=b(s)\}}\,dL_s^x$$

It follows,

$$\int_0^t \int_{I\!R} F_x(s,\,x)\,dL_s^x \;=\; -\int_0^t F_{xx}(s,B_s)\,I(B_s\,\neq\,b(s))\,ds - \int_0^t \Big(F_x(s,b_s+) - F_x(s,b_s-)\Big)\,d_sL_s^{b(s)}$$

To conclude we need to recall the extended definition of the local time to the borelian curves due to Eisenbaum [5]:

$$L_t^{b(\cdot)} \;=\; \int_0^t \int_{I\!R} 1_{(-\infty,\,b(s))}(x)\,dL_s^x$$

It follows that $d_t\,L_t^{b(\cdot)} \;=\; d_t\,L_t^{b(t)}$.
We suggest another method based on Theorem 6.3.1:

$$\frac{1}{4\,\varepsilon}\int_0^t \Big\{F_x(s,B_s+\varepsilon) \;-\; F_x(s,B_s-\varepsilon)\Big\}\,d\,s = I_\varepsilon + J_\varepsilon$$

$$I_\varepsilon := \frac{1}{4\,\varepsilon}\int_0^t 1_{(-\varepsilon<B_s-b_s<+\varepsilon)^c}\Big\{F_x(s,B_s+\varepsilon) \;-\; F_x(s,B_s-\varepsilon)\Big\}\,ds$$

$$J_\varepsilon := \frac{1}{4\,\varepsilon}\int_0^t 1_{(-\varepsilon<B_s-b_s<+\varepsilon)}\Big\{F_x(s,B_s+\varepsilon) \;-\; F_x(s,B_s-\varepsilon)\Big\}\,ds\,.$$

We have the following (see ([11]) for details):

$$\lim_{\varepsilon\downarrow 0} I_\varepsilon \;=\; \frac{1}{2}\int_0^t 1_{(B_s\,\neq\,b_s)}\,F_{xx}(s,B_s)\,ds$$

and

$$\lim_{\varepsilon\downarrow 0} J_\varepsilon \;=\; \frac{1}{2}\int_0^t 1_{(B_s\,=\,b_s)}\Big\{F_x(s,B_s+) \;-\; F_x(s,B_s-)\Big\}\,d\,L_s^b\,.$$

□

Remark 4.1 1. For an extension of Theorem 6.4.3 to continuous semimartingales see the paper by Peskir [12]. Independantly, see also Elworthy et al [4].
2. A particular case have been obtained by Jacka (see Section 5.2 in [8]).

Corollary 4.1 (the discrete case of $b(\cdot)$ reduced to $\{a_1,\cdots,a_n\}$) *Let $B=(B_t)_{t\geq 0}$ be a standard Brownian motion and let $a_1 < a_2 < \cdots < a_n$ be real numbers, and denote $D=\{a_1,\cdots,a_n\}$. Suppose that a function $F: I\!R \to I\!R$ is continuous and F_x and F_{xx} exist and are continuous on $I\!R\setminus D$, and the limits*

$$F_x(a_k\pm) := \lim_{x\to a_k\pm} F_x(x) \quad F_{xx}(a_k\pm) := \lim_{x\to a_k\pm} F_{xx}(x)$$

exist and are finite.
Then we have:

$$F(B_t) = F(B_0) + \int_0^t F_x(B_s)\,dB_s + \frac{1}{2}\int_0^t F_{xx}(B_s)\;ds + \frac{1}{2}\sum_{i=1}^n \Big\{F_x(a_i+) \;-\; F_x(a_i-)\Big\}\,L_t^{a_i}$$

Remark 4.2 Corollary 6.4.1 is actually Problem 6.24 given in Karatzas and Shreve [9].

References

[1] N. Bouleau *and* M. Yor, *Sur la variation quadratique des temps locaux de certaines semimartingales.* C. R. Acad. Sci. Paris Sér. I Math **292** (1981), 491–494.

[2] A. S. Cherny, *Principal values of the integral functionals of Brownian motion: Existence, continuity and extension of Itô's formula.* Sém. Probab. 35, Lecture Notes in Math. **1755** (2001), 348–370.

[3] R. J. Chitashvili *and* M. G. Mania, *Decomposition of the maximum of semimartingales and generalized Itô's formula.* New trends in probability and statistics, Vol. **1** (Bakuriani, 1990), 301–350.

[4] K. D. Elworthy, A. Truman *and* H. Z. Zhao, *A generalized Itô formula and asymptotics of heat equations with caustics in one-dimension.* Preprint (http://mahzmac.lboro.ac.uk/%7emahz/papers.html).

[5] N. Eisenbaum, *Integration with respect to local time.* Potential Anal. **13** (2000), 303–328.

[6] N. Eisenbaum, *On local times of a symmetric stable process as a doubly indexed process.* Bernoulli **6** (2000), 871–886.

[7] H. Föllmer, P. Protter *and* A. N. Shiryayev, *Quadratic covariation and an extension of Itô's formula.* Bernoulli **1** (1995), 149–169.

[8] S. D. Jacka, *Local times, optimal stopping and semimartingales.* Ann. Probab. **21** (1993), 329–339.

[9] I. Karatzas *and* S. E. Shreve, *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York, (1991).

[10] S. F. Nasyrov, *On local times for functions and stochastic processes: II.* Theory Probab. Appl. **41** (1996), 275–287.

[11] J. L. Pedersen *and* G. Peskir, *On nonlinear integral equations arising in problems of optimal stopping.* Research Report No. **426** (2002) Dept. Theoret. Statist. Aarhus (17 pp). *Proc. Funct. Anal.* **VII** (Dubrovnik 2001), *Various Publ. Ser.* **46**, (2002), 159–175.

[12] G. Peskir, *A change-of-variable formula with local time on curves.* Research Report No. **428** (2002), Dept. Theoret. Statist. Aarhus (17 pp).

[13] Ph. Protter *and* J. San Martin, *General change of variable formulas for semimartingales in one and finite dimensions.* Probab. Theory Related Fields **97** (1993), 363–381.

[14] J. B. Walsh, *Some remarks on* $A(t, B_t)$. Sém. Probab. 27, Lecture Notes in Math. **1557** (1993), 173–176.