SUBHARMONICITY OF HIGHER DIMENSIONAL EXPONENTIAL TRANSFORM

VLADIMIR TKACHEV

1. The exponential transform [2] can be viewed as a potential depending on a domain in $\mathbb{R}^n$, or more generally on a measure having a *density* function $\rho(x)$ (with compact support) in the range $0 \le \rho \le 1$:

$$E_\rho(x) = \exp\left[-\frac{2}{\omega_n}\int \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n}\right].$$

The two-dimensional (polarized) version has appeared in operator theory, as a determinantal-characteristic function of certain close to normal operators [1], [3], and has previously been studied and proved to be useful within operator theory, moment problems and other problems of domain identification, and for proving regularity of free boundaries.

For all $n \ge 3$ it is known that E_ρ is a subharmonic function; on the other hand, for $n = 2$ the function $\ln(1 - E_\rho)$ is known to be subharmonic, which is a stronger statement [2].

Our main result extends the mentioned subharmonicity in dimension $n \ge 3$ thereby answering in affirmative a recent conjecture posed in [2]:

Theorem 1. *Let $E_\rho(x)$ be the exponential transform of a density $\rho \not\equiv 0$. Then the function*

$$(1) \qquad \begin{cases} \ln(1-E_\rho), & \text{if} \quad n=2, \\ \frac{1}{n-2}(1-E_\rho)^{(n-2)/n}, & \text{if} \quad n \ge 3, \end{cases}$$

is subharmonic outside $\operatorname{supp}\rho$.

In fact, we show that a stronger version holds. Namely, let $\mathcal{M}_n(t)$ denotes the *profile* function, i.e. the solution of the following problem:

$$(2) \qquad \mathcal{M}_n'(t) = 1 - \mathcal{M}_n^{2/n}(t), \qquad \mathcal{M}(0) = 0.$$

Theorem 2. *For $n \ge 2$ let ρ be a density function and*

$$(3) \qquad V_\rho(x) = ⨍ \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n} \equiv \frac{n}{\omega_n}\int \frac{\rho(\zeta)d\zeta}{|x-\zeta|^n}.$$

Then the function

$$(4) \qquad \begin{cases} \log \mathcal{M}_2(V_\rho(x)), & \text{if} \quad n=2 \\ [\mathcal{M}_n(V_\rho(x))]^{(n-2)/n}, & \text{if} \quad n \ne 2 \end{cases}$$

is subharmonic outside the support of ρ.

The latter property is sharp in the sense that $\mathcal{E}_{\widehat{\rho}}(x)$ is a harmonic function (in $\mathbb{R}^n \setminus \operatorname{supp}\widehat{\rho}$) if $\widehat{\rho} = \chi_B$ is the characteristic function of a Euclidean ball.

Our key technical result is the following Cauchy-type inequality (for bounded densities). Let $0 \notin \operatorname{supp}\rho$, and $0 \le \rho \le 1$. Then

$$(5) \qquad \left(⨍_{\mathbb{R}^n} \frac{x_1\rho(x)}{|x|^n}\,dx\right)^2 \le \mathcal{M}_n\left(⨍_{\mathbb{R}^n} \frac{\rho(x)}{|x|^n}\,dx\right)⨍_{\mathbb{R}^n} \frac{\rho(x)}{|x|^{n-2}}\,dx,$$

This research was supported by grant RFBR no. 03-01-00304 and by Göran Gustafsson Foundation.

with equality case when $\widehat{\rho}$ is the characteristic function of a Euclidean ball. Noteworthy, that for $n \geq 3$, inequality (5) can be interpreted as a pointwise estimate on the Coulomb potential

$$|\nabla U_\rho(x)|^2 \leq \mathcal{M}_n[V_\rho(x)]U_\rho(x) < U_\rho(x), \qquad x \notin \operatorname{supp} \rho.$$

As a useful application of the above theorems we mention the following analogue of the well-known Ahlfors and Beurling estimate of the logarithmic capacity of a planar domain (this inequality was a start point of Gustafsson and Putinar [2] to conjecture subharmonicity in Theorem 1).

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain satisfying $\Omega = \operatorname{int} \overline{\Omega}$. Then*

$$(n-2)2^{(n-2)/n}|\omega_n|^{2/n}|\Omega|^{(n-2)/n} \leq \operatorname{cap}(\overline{\Omega}).$$

2. We also discuss structure properties of the profile function $\mathcal{M}_n$ in more detail. This higher transcendental function, apart of its appearance in the above theorems, admits also number-theoretical applications (e.g., in connection with the Euler-Mascheroni constant γ).

We recall that a function $f(x)$ defined on $[0;+\infty)$ is said to be *completely monotonic* if $(-1)^k f^{(k)}(x) \geq 0$, for all $x \in \mathbb{R}^+$. Let

$$\phi_\alpha(t) := 1 - M_{2/\alpha}\left(-\frac{1}{\alpha}\ln t\right).$$

Theorem 3. *]For any $\alpha > 0$ the function $\phi_\alpha(t)$ admits an analytic continuation on $(-\epsilon, 1)$ with some $\epsilon > 0$ depending on α. In particular, the corresponding Taylor series at $t = 0$ are*

$$\phi_\alpha(t) = \sum_{k=1}^{\infty} \sigma_k (\gamma_\alpha t)^k, \tag{6}$$

where

$$\gamma(\alpha) = \frac{1}{\alpha}\exp\biggl(-\int_0^1 \frac{1 - x^{\frac{1-\alpha}{\alpha}}}{1-x}\,dx\biggr),$$

and σ_k are the coefficients defined by the following recurrence

$$\sigma_1 := 1, \quad \sigma_k = \frac{1}{k(k-1)}\sum_{\nu=1}^{k-1} \sigma_\nu \sigma_{k-\nu}[(1+\alpha)\nu - \alpha k]\nu. \tag{7}$$

Moreover, if $\alpha \in (0,1)$ then $\sigma_k > 0$ for all $k \geq 1$ and series (6) converges in $(-1,1)$. For all $0 < \alpha < 1$, $\phi_\alpha(t)$ is a strictly increasing convex function in $(-\infty, 1)$.

We have the following explicit representation of the profile function.

Corollary 2. *Let $n \geq 2$ be an integer. Then $1 - \mathcal{M}_n(w)$ is a completely monotonic function, and*

$$1 - \mathcal{M}_n(x) = \sum_{k=1}^{\infty} a_k e^{-2kx/n},$$

where $a_k = \sigma_k \gamma_{2/n}^k > 0$ and the series converges for all $x \geq 0$.

References

[1] R. W. Carey and J. D. Pincus, *An exponential formula for determining functions*, Indiana Univ. Math. J., **23** (1974), 1031–1042.

[2] B. Gustafsson and M. Putinar, *The exponential transform: a renormalized Riesz potential at critical exponent.* Ind. Univ. Math. J., **52** (2003), no. 3, 527–568.

[3] J. D. Pincus and J. Rovnyak, *A representation formula for determining functions*, Proc. Amer. Math. Soc., **22** (1969), 498–502.

Volgograd State University
E-mail address: vladimir.tkachev@volsu.ru
URL: http://www.math.kth.se/~tkatchev