SEPARATION OF VARIABLES FOR NONLINEAR WAVE EQUATION IN POLAR COORDINATES

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The two-dimensional nonlinear wave equation for a potential $\phi(x_1, x_2, t)$

$$-\kappa\phi_{tt} + \triangle\phi + \frac{\alpha}{2}\left(\nabla\phi\bullet\nabla\phi\right)_t + \frac{\beta}{2}\left(\phi_t^2\right)_t = 0 \tag{1}$$

describes, in particular, the long surface water waves and the two-dimensional waves in an isentropic gas flow for the non-dissipative case.

The aim of this paper is to present a series of its solutions in the polar coordinates (θ, r). These solutions describe periodic waves found with the same accuracy as equation (1) is derived. The linear versions of these solutions were studied and used in a series of classic books (see [1] and [2]). The presented solutions give nonlinear corrections to these well-known classical linear solutions.

Given in the polar coordinates, equation (1) for $\varphi(r, \theta, t) = \phi(x_1, x_2, t)$ can be written as follows:

$$-\kappa\varphi_{tt} + \frac{1}{r}\varphi_r + \varphi_{rr} + \frac{1}{r^2}\varphi_{\theta\theta} + \frac{\alpha}{2}\left(\varphi_r^2 + \frac{1}{r^2}\varphi_\theta^2\right)_t + \frac{\beta}{2}\left(\varphi_t^2\right)_t = 0 \tag{2}$$

The potential $\varphi(r, \theta, t)$ is assumed to be regular in the vicinity of the origin of the coordinates and expanded in Fourier series in t:

$$\varphi(r, \theta, t) = \varepsilon\left[S_1(r, \theta)\sin(\omega t) + C_1(r, \theta)\cos(\omega t)\right] + \varepsilon^2\left[S_2(r, \theta)\sin(2\omega t) + C_2(r, \theta)\cos(2\omega t)\right] + ... \tag{3}$$

Then the functions $S_1(r, \theta)$ and $C_1(r, \theta)$ satisfy the Helmholz equation $Z_{rr} + \frac{1}{r}Z_r + \frac{1}{r^2}Z_{\theta\theta} + \kappa\omega^2 Z = 0$, and their Fourier expansions with respect θ can be written as follows:

$$S_1(r, \theta) = a_0 J_0(kr) + J_1(kr)(a_1\sin\theta + b_1\cos\theta) + ... + J_j(kr)(a_j\sin j\theta + b_j\cos j\theta) + ..., \tag{4}$$

$$C_1(r, \theta) = c_0 J_0(kr) + J_1(kr)(c_1\sin\theta + d_1\cos\theta) + ... + J_j(kr)(c_j\sin j\theta + d_j\cos j\theta) + ..., \tag{5}$$

The functions $S_2(r, \theta)$ and $C_2(r, \theta)$ can also be expanded in the Fourier series with respect θ:

$$S_2(r, \theta) = M_0(r) + \textstyle\sum_{j=1}^{\infty}\left(M_j(r)\sin j\theta + N_j(r)\cos j\theta\right), \quad C_2(r, \theta) = P_0(r) + \textstyle\sum_{j=1}^{\infty}\left(P_j(r)\sin j\theta + Q_j(r)\cos j\theta\right). \tag{6}$$

We seek functions $M_j(r)$, $N_j(r)$, $P_j(r)$, and $Q_j(r)$ in the form

$$R_{00}J_0^2 + R_{01}J_0J_1 + R_{11}J_1^2, \tag{7}$$

where R_{il} are polynomials of r^{-1} and r with unknown coefficients

$$R_{il} = \textstyle\sum_{k=-n}^{n} C_k^{il} r^k. \tag{8}$$

However, if a_j, b_j, c_j, and d_j are fixed, substituting (3), (4) and (5) in (2), we obtain **overdetermined** systems of linear algebraic equations for C_k^{il}.

The key point of consideration is that these **overdetermined** systems are compatible. This allow us to construct explicit expressions for the functions S_2 and C_2 of the form (6) and (7).

Representation of $\varphi(r, \theta, t)$ by expansions (6) can be considered as a generalization of the method of separation of variables for the nonlinear equation. Possibility of its generalization for other special coordinates will be discussed in the talk.

The similar approach was used in [3] and [4] for describing the long periodic water waves on a slope in the high-order shallow water approximation and in [5] and [6] for finding the first terms of expansion (6).

References

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