Quasiconvex pair: necessary and sufficient condition for weak lower semicontinuity of some integral functionals

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The property of weak lower semicontinuity of the functional $I(u)$, defined on a Banach space X:

$$I(u) \leq \liminf_{n\to\infty} I(u_n) \quad \text{for every sequence} \quad u_n \rightharpoonup u$$

plays a crucial role for application of direct methods of Calculus of Variations.

In many applications one faces the problem of minimizing the functionals of the following type:

$$I(u,\chi) = \int_\Omega \Big\{ \chi(x) F^+(x, u(x), \nabla u(x)) + (1-\chi(x)) F^-(x, u(x), \nabla u(x)) \Big\} dx, \tag{1}$$

where

$$u \in W^{1,p}(\Omega;\mathbb{R}^m), \quad p \in (1,\infty), \quad \chi \in Z, \quad Z = \{\chi \in L_\infty(\Omega) : 0 \leq \chi(x) \leq 1 \text{ a.e. } x \in \Omega\}.$$

We say that the functions $F^\pm$ generate a *quasiconvex pair*, if for every $M \in \mathbb{M}^{m\times n},\ u \in \mathbb{R}^m,\ x \in \Omega$, for all subdomains $\omega \subset \Omega$ and all functions $\phi \in W_0^{1,p}(\omega;\mathbb{R}^m),\ \chi \in Z$ the following inequality holds:

$$\begin{aligned} F^+(M,u,x)\int_\omega \chi(y)dy + F^-(M,u,x)\int_\omega (1-\chi(y))dy \leq \\ \leq \int_\omega \Big\{ \chi(y) F^+(M+\nabla\phi(y), u, x) + (1-\chi(y)) F^-(M+\nabla\phi(y), u, x) \Big\} dy. \end{aligned} \tag{2}$$

The criterion for weak lower semicontinuity:

$$I(u,\chi) \leq \liminf_n I(u_n,\chi_n)$$

for all sequences

$$W^{1,p}(\Omega;\mathbb{R}^m) \ni u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m), \quad Z \ni \chi_n \rightharpoonup \chi \text{ in } L_p(\Omega) \tag{3}$$

for the functional of the type (1) is established:

Theorem 0.1 *The functional (1) is weakly l.s.c. with respect to the convergence (3) if and only if the functions $F^\pm$ generate a quasiconvex pair in the sense of the definition (2).*

This criterion has been obtained in [1] for the case of Lipschitz functions $F^\pm$. Here we generalize this result for arbitrary Caratheodory functions $F^\pm$, satisfying well known growth conditions.

In this case for proving the sufficiency part, Young measure technique is used: a definition of Young measure, satisfying **(W)** property is introduced: it is a Young measure ν generated by the sequence of pairs $(\nabla u_k, \chi_k) \in L_p(\Omega) \times L_p(\Omega)$, such that $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$, and $\chi_k \rightharpoonup \chi$ in $L_p(\Omega)$. It turns out, that these Young measures satisfy the analogue of Jensen inequality for all quasiconvex pairs of functions. Based on this inequality, the weak lower semicontinuity can be proved.

References

[1] V. G. Osmolovskii (2003) "Weak lower semicontinuity criterion of the energy functional of two phase media" (in Russian), *Problemy Mat. Analisa*, **26**, 215-254.