On the Finite Topology of an R Module. Applications to Corings

Miodrag Cristian Iovanov
Department of Algebra, Faculty of Mathematics, University of Bucharest
Academiei 14, Bucharest, Romania

Abstract.In this paper we study the properties of the finite topology on the dual of a module over arbitrary rings. We aim to give conditions when certain properties of the field case are can be still found here.

1 Introduction and preliminaries

Let R be an arbitrary (noncommutative) ring. We will use the notations $\mathrm{Hom}_R(M,N)$ for the set of R module morphisms from M to N for right modules M,N and ${}_R\mathrm{Hom}(M,N)$ respectively for left modules M,N. Also we use $M^*=\mathrm{Hom}_R(M,R)$ for any right module M and ${}^*M={}_R\mathrm{Hom}(M,R)$ for a left module M.

Given two right R modules M and N, recall that the finite topology on $\mathrm{Hom}_R(M,N)$ is the linear topology for which a basis of open neighbourhoods for 0 is given by the sets $\{f\in \mathrm{Hom}_R(M,N) \mid f(x_i)=0, \forall\, i\in\{1,\dots,n\}\}$, for all finite sets $\{x_1,\dots,x_n\}\subseteq M$. This is actually the topology induced on $\mathrm{Hom}_R(M,N)$ from $\mathrm{Hom}_{Set}(M,N)=N^M$ which is a product of topological spaces, where N is the topological discrete space on the set N. For an arbitrary set $X\subseteq M$ we denote by $X^{\perp}=\{f\in \mathrm{Hom}_R(M,N)\mid f|_X=0\}$. Denoting by $<X>_R$ the R submodule generated by X, we obviously have $(<X>_R)^{\perp}=X^{\perp}$, so we will work with finitely generated submodules $F\leq M$ and the basis of open neighbourhoods $\{F^{\perp}\mid F\leq M \,\text{finitely generated}\}$. Also for left R modules X and Y and $U\leq X$ a submodule of X we will denote $U^{\perp}_{{}_R\mathrm{Hom}(M,N)}$ or simply $U^{\perp}=\{g\in {}_R\mathrm{Hom}(X,Y)\mid g|_X=0\}$ when there is no danger of confusion. If $W\leq \mathrm{Hom}_R(M,N)$ is a subgroup with M and N left R modules we denote $W^{\perp}=\{x\in N\mid f(x)=0, \forall\, f\in W\}$. If N is an R bimodule then we consider the left R module structure on $\mathrm{Hom}_R(M,N)$ given by $(r\cdot f)(x)=rf(x)$, for all $x\in M$, $f\in \mathrm{Hom}_R(M,N)$, $r\in R$. If W is a (left) submodule in $\mathrm{Hom}_R(M,N)$, then $W^{\perp}$ is a (right) submodule of M.

For any right module M we denote by Φ_M the right R modules morphism

$$M \xrightarrow{\Phi_M} {}^*(M^*)$$

defined by $\Phi_M(m)(f)=f(m)$, for all $f\in M^*$ and all $m\in M$. Then Φ is a functorial morphism from $id_{\mathcal{M}_R}$ to the functor ${}^*((-)^*)$.

Over a field, there is a series of properties involving the orthogonal $F^{\perp}$ for a vector space V and its dual V^* which we will state in a more general setting.

Proposition 1.1 *Let M, N be R modules.*
(i) If $X \subseteq Y$ are submodules of M then $Y^{\perp} \leq X^{\perp}$.
(ii) If $U \subseteq V$ are subgroups of $\mathrm{Hom}_R(M,N)$ then $V^{\perp} \leq U^{\perp}$.

Lemma 1.2 *For M, N right R modules we have:*
(i) If $X \leq M$ is a submodule of M then $(X^{\perp})^{\perp} \supseteq X$. If N is an injective cogenerator of $\mathcal{M}_R$ then the equality $(X^{\perp})^{\perp} = X$ holds.
(ii) If $Y \leq \mathrm{Hom}_R(M,N)$ is a (left) submodule of $\mathrm{Hom}_R(M,N)$ then $(Y^{\perp})^{\perp} \supseteq \overline{Y}$ ($\overline{Y}$ is the closure of Y in $\mathrm{Hom}_R(M,N)$). If $N = R$ and R is a left PF ring (${}_RR$ is injective and a cogenerator of ${}_R\mathcal{M}$) then the equality $(Y^{\perp})^{\perp} = \overline{Y}$ holds for all modules M and (left) submodules $Y \leq M^*$.

Proof. (i) If $x \in X$ then take $f \in X^{\perp}$; then $f(x) = 0$ as $f|_X = 0$. We get that $f(x) = 0, \forall f \in X^{\perp}$ so $x \in (X^{\perp})^{\perp}$.
Suppose now N is an injective cogenerator of $\mathcal{M}_R$ and take $x \in (X^{\perp})^{\perp}$. If $x \notin X$ then there is $f : M/X \longrightarrow N$ such that $f(\hat{x}) \neq 0$ ($\hat{x}$ is the immage of x in M/X via the canonic morphism $\pi : M \longrightarrow M/X$). Then there is $g = f \circ \pi$, $g \in \mathrm{Hom}_R(M,N)$ such that $g|_X = 0$ ($g \in X^{\perp}$) and $g(x) \neq 0$, showing that $x \notin (X^{\perp})^{\perp}$, a contradiction.
(ii) Let $f \in \overline{Y}$ and take $x \in Y^{\perp}$. Then there is $g \in Y$ such that $f(x) = g(x)$. But $g(x) = 0$ because $x \in Y^{\perp}$ so $f(x) = 0$. Thus $f|_{Y^{\perp}} = 0$ and $f \in (Y^{\perp})^{\perp}$.
For the converse, first we see that ${}_RR$ implies that for all finitely generated right R modules F we have that $F \xrightarrow{\Phi_F} {}^*(F^*)$ is an epimorphism. Take $\pi : P = R^n \longrightarrow F$ an epimorphism in $\mathcal{M}_R$. Then we have a monomorphism $0 \longrightarrow P^* \longrightarrow F^*$ in ${}_R\mathcal{M}$, and as ${}_RR$ is injective we obtain an epimorphism of right modules ${}^*(P^*) \xrightarrow{{}^*(p^*)} {}^*(F^*) \longrightarrow 0$. Because Φ is a functorial morphism then we have the commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\pi} & F & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Phi_P} & & \downarrow{\scriptstyle \Phi_F} & & \\ {}^*(P^*) & \xrightarrow[{}^*(\pi^*)]{} & {}^*(F^*) & \longrightarrow & 0 \end{array}$$

and diagram

$$\begin{array}{ccccc} P & \xrightarrow{\pi} & F & \longrightarrow & 0 \\ \downarrow{\scriptstyle \Phi_P} & & \downarrow{\scriptstyle \Phi_F} & & \\ {}^*(P^*) & \xrightarrow[{}^*(\pi^*)]{} & {}^*(F^*) & \longrightarrow & 0 \end{array}$$

showing that Φ_F is surjective, as $\Phi_P = \Phi_{R^n}$ is an isomorphism. Now to prove the desired equality, take $f \in (Y^{\perp})^{\perp}$, $(f_i)_{i \in I}$ a family of generators of the left R module Y, and $F < M$ a finitely generated submodule of M. Then $f_i|M \in F^*$ and if $f|F \notin {}_R < f_i|_F \mid i \in I >$ then as ${}_RR$ is an injective cogenerator of ${}_R\mathcal{M}$ we can find a morphism of left R modules $\phi : F^* \longrightarrow R$ such that $\phi(f_i) = 0$, $\forall i \in I$ and $\phi(f) \neq 0$. But as Φ_F is surjective, we can then find $x \in F$ such that $\phi = \Phi(x)$ and then $f_i(x)\Phi(x)(f_i) = \phi(f_i) = 0$, $\forall i \in I$, showing that $x \in Y^\perp$ and $f(x) = \Phi(x)(f) = \phi(f) \neq 0$ which contradicts the fact that f belongs to $(Y^\perp)^\perp$. Thus we must have $f|_F \in {}_R < f_i|_F \mid i \in I >$ so there is $(r_i)_{i\in I}$ a family of finite support such that $f|_F = \sum\limits_{i\in I} r_i(f_i|_F) = (\sum\limits_{i\in I} r_i f_i)|_F$. This last relation shows that $f \in \overline{Y}$. □

Proposition 1.3 *Let M be a right R module.*
(i) If $X \leq M$ then we have $((X^\perp)^\perp)^\perp = X^\perp$ and $X^\perp$ is closed.
(ii) If $Y \leq \mathrm{Hom}_R(M,N)$ then $((Y^\perp)^\perp)^\perp = Y^\perp$.

Proof. "$\subseteq$" from (i) and (ii) follow from Proposition 1.1 and Lemma 1.2.
(i) "$\supseteq$" Let $f \in X^\perp$. Take $x \in (X^\perp)^\perp$; then $f(x) = 0$ so $f \in ((X^\perp)^\perp)^\perp$. To show that $X^\perp$ is closed take $f \in \overline{X^\perp}$ and $x \in X$. Then there is $g \in X^\perp$ such that $g(x) = f(x)$ so $f(x) = 0$ $(x \in X)$. We obtain that $f|_X = 0$ so $f \in X^\perp$.
(ii) "$\supseteq$" Let $x \in Y^\perp$. If $f \in (Y^\perp)^\perp$ then $f|_{Y^\perp} = 0$ so $f(x) = 0$ showing that $x \in ((Y^\perp)^\perp)^\perp$. □

Proposition 1.4 *Let M, N be right R modules and $(X_i)_{i\in I}$ a family of submodules of M. Then*
(i) $(\sum\limits_{i\in I} X_i)^\perp = \bigcap\limits_{i\in I} X_i^\perp$.
(ii) $(\bigcap\limits_{i\in I} X_i)^\perp \supseteq \sum\limits_{i\in I} X_i^\perp$. If I is finite and N is injective then equality holds.

Proof. (i) $f \in (\sum\limits_{i\in I} X_i)^\perp \Leftrightarrow f|_{\sum_{i\in I} X_i} = 0 \Leftrightarrow f|_{X_i} = 0, \forall i \in I \Leftrightarrow f \in X_i^\perp, \forall i \in I \Leftrightarrow f \in \bigcap\limits_{i\in I} X_i^\perp$.
(ii) "$\supseteq$" is obvious, for Proposition 1.1 shows that $X_i^\perp \subseteq \bigcap\limits_{j\in I} X_j{}^\perp$, $\forall i \in I$. For the converse it is enough to prove the equality for two submodules X, Y of M. Denote $\pi : M \longrightarrow M/X\cap Y$, $p : M \longrightarrow M/X$, $q : M \longrightarrow M/Y$ the canonical morphisms. If $f \in \mathrm{Hom}_R(M,N)$ such that $f|_{X\cap Y} = 0$ then denote $\overline{f} : M/X\cap Y \longrightarrow N$ the factorisation of f ($f = \overline{f} \circ \pi$) and $i : M/X\cap Y \longrightarrow M/X \oplus M/Y$ the injection $i(\pi(x)) = (p(x), q(x))$, $\forall x \in M$. Then the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \dfrac{M}{X\cap Y} & \xrightarrow{\ i\ } & \dfrac{M}{X} \oplus \dfrac{M}{Y} \\ & & \Big\downarrow \overline{f} & \swarrow h = \overline{u} \oplus \overline{v} & \\ & & N & & \end{array}$$

is completed commutatively by h. Then $h = \overline{u} \oplus \overline{v}$, with $\overline{u} \in \mathrm{Hom}_R(M/X, N)$ and $\mathrm{Hom}_R(M/Y, N)$, such that $h(p(x), q(x)) = \overline{u}(p(x)) + \overline{v}(q(x))$. Taking $u = \overline{u} \circ p$ and $v = \overline{v} \circ q$ we have $u \in X^\perp$, $v \in Y^\perp$ and $f(x) = \overline{f}(\pi(x)) = h(i(\pi(x))) = h(p(x), q(x)) = \overline{u}(p(x)) + \overline{v}(q(x)) = u(x) + v(x)$, $\forall x \in M$, so $f \in X^\perp + Y^\perp$. □

Proposition 1.5 *Let M, N be right R modules and $(Y_i)_{i\in I}$ a family of submodules of* $\mathrm{Hom}_R(M,N)$. *Then:*
(i) $(\sum_{i\in I} Y_i)^{\perp} = \bigcap_{i\in I} Y_i^{\perp}$.
(ii) $(\bigcap_{i\in I} Y_i)^{\perp} \supseteq \sum_{i\in I} Y_i^{\perp}$. *If $N = R$ and R is a PF ring (both left and right PF), I is a finite set and Y_i are closed subsets of $M^* = \mathrm{Hom}_R(M,R)$ then the equality holds:* $(\bigcap_{i\in I} Y_i)^{\perp} = \sum_{i\in I} Y_i^{\perp}$.

Proof. (i) Obvious.
(ii) "$\supseteq$" similar to (ii)"$\supseteq$" of the previous proposition. For the converse inclusion, take X, Y submodules of M^*. Then

$$\begin{aligned} X^{\perp} + Y^{\perp} &= ((X^{\perp} + Y^{\perp})^{\perp})^{\perp} \quad (\text{from Lemma 1.2} : R \,\text{is right PF}) \\ &= ((X^{\perp})^{\perp} \cap (Y^{\perp})^{\perp})^{\perp} \quad (\text{from Proposition 1.4}) \\ &= (X \cap Y)^{\perp} \quad (\text{Lemma 1.2 } : X, Y \,\text{are closed and}\, R \,\text{is left PF}) \end{aligned}$$

so the conclusion follows for finitely many submodules of M^*. □

Example 1.6 *(i) We show that the equality in Proposition 1.4 does not hold for infinite sets. Let V be a infinite dimensional space with a countable basis indexed by the set of natural numbers: $(e_n)_{n\in\mathbf{N}}$. Put $V_n =< e_k \mid k \geq n >$. Then we can easily see that $\bigcap_{n\in\mathbf{N}} V_n = 0$ so $(\bigcap_{n\in\mathbf{N}} V_n = 0)^{\perp} = V^*$. Let $f \in V^*$ be the function equal to 1 on all the e_n-s. Then as $V_n^{\perp} < V_m^{\perp}, \forall\, n < m$, we have that $f \in \sum_{n\in\mathbf{N}} V_n^{\perp} \Leftrightarrow \exists\, n \in \mathbf{N}$ such that $f \in V_n^{\perp}$ which is imposible as $f(e_n) = 0, \forall\, n$. We obtain $\bigcap_{n\in\mathbf{N}} V_n \supset \sum_{n\in\mathbf{N}} V_n^{\perp}$ a strict inclusion.*
(ii)

2 The Finite Topology vs PF Rings

If R is a ring then we have $R^* = \mathrm{Hom}_R(R,R) \simeq {}_RR$. So we can identify R submodules of the right dual of R with left ideals of R