On pseudo-spherical congruencies in E^4

Gorkavyy V.A.
B.Verkin Institute for Low Temperature Physics, Kharkiv, Ukraine
E-mail: gorkaviy@ilt.kharkov.ua

Let $F^2, \tilde{F}^2$ be regular two-dimensional surfaces in four-dimensional Euclidean space E^4. *A line congruence* $\psi : F^2 \to \tilde{F}^2$ is a diffeomorphism which possesses the following bitangency property: for each point $P \in F^2$ the straight line joining P with $\psi(P) = \tilde{P} \in \tilde{F}^2$ is a common tangent line for F^2 and $\tilde{F}^2$. The line congruence $\psi : F^2 \to \tilde{F}^2$ is said to be *pseudo-spherical* if it satisfies two additional conditions:

B1) the distance between corresponding points $P \in F^2$ and $\tilde{P} \in \tilde{F}^2$ is equal to a non-zero constant independent of P, $|P\tilde{P}| \equiv l_0 \neq 0$;

B2) the angle between planes tangent to F^2 and $\tilde{F}^2$ at corresponding points is equal to a non-zero constant independent of P, $\angle(T_P F^2, T_{\tilde{P}}\tilde{F}^2) \equiv \omega_0 \neq 0$.
This construction corresponds to the classical definition of pseudo-spherical congruencies for n-dimensional submanifolds in $(2n-1)$-dimensional Euclidean space, see [1], [2].

We prove that if two surfaces F^2, $\tilde{F}^2$ in E^4 are connected by a pseudo-spherical congruence than in the general case F^2 and $\tilde{F}^2$ are of constant negative Gauss curvature $K = -sin^2\omega_0/l^2$ – a similar statement holds for n-dimensional submanifolds in E^{2n-1} [2]. On the other hand, contrary to the classical case, a pseudo-spherical surface in E^4 admits at most two pseudo-spherical congruencies. Besides we completely describe pseudo-spherical surfaces in E^4 which admit pseudo-spherical congruencies with $\omega_0 = \pi/2$ (Bianchi congruencies). From analytic point of view, such surfaces are described by solutions $\{\varphi(u,v), P(u,v), Q(u,v)\}$ of the following system of p.d.e.:

$$\partial_{uu}e^{2\varphi} + \partial_{vv}e^{-2\varphi} + 2(PQ+1) = 0$$

$$\partial_u P - \partial_u\varphi Q e^{2\varphi} = 0, \quad \partial_v Q + \partial_v \varphi P e^{-2\varphi} = 0,$$

whereas Bianchi congruences may be interpreted as the following transformation of solutions:

$$\{\varphi(u,v), P(u,v), Q(u,v)\} \to \{-\varphi(-v,-u), -Q(-v,-u), -P(-v,-u)\}.$$

These results complement investigations realised by Yu.Aminov and A.Sym in [3].

References

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