A unique encoding of spatial graphs in 3-space

V. A. Kurlin

We present the method of 3-page embeddings, that reduces completely the isotopy classification of spatial graphs in $\mathbb{R}^3$ to a pure algebraic problem.

We consider non-oriented finite graphs up to homeomorphism. Self-loops and multiple edges are allowed, vertices of degree 1 are excluded.

Let G be a finite graph. *A spatial* (or *knotted*) graph is a subset $G \subset \mathbb{R}^3$, homeomorphic to G. *An isotopy* between two spatial graphs $G, H \subset \mathbb{R}^3$ is a continuous family of homeomorphisms $\phi_t : \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,1]$, such that $\phi_0 = \mathrm{id}$ and $\phi_1(G) = H$. If at each moment $t \in [0,1]$ of the isotopy a neighbourhood of every vertex in the graph $\phi_t(G) \subset \mathbb{R}^3$ lies in a (non-constant) plane, then ϕ_t is called *a rigid isotopy*. Otherwise ϕ_t is *non-rigid*.

The spatial 4-valent graph G of Fig. 1 is non-rigid isotopic to the unknotted figure eight graph ∞, but these graphs are not rigid isotopic. The method of 3-page embeddings was investigated by Dynnikov for non-oriented links [1]. We have extended this apporach to arbitrary spatial graphs [2, 3]. For each $n \geq 2$, we construct finitely presented semigroups RSG_n and NSG_n generated by the letters $\{\ a_i,\ b_i,\ c_i,\ d_i,\ x_{m,i} \mid i = 0,1,2;\ 3 \leq m \leq n\}$. Explicit presentations of RSG_n and NSG_n could be found in [3, p. 2].

Theorem [3]. The center of RSG_n (resp. NSG_n) encodes uniquely all spatial graphs with vertices of degree $\leq n$ up to rigid (resp. non-rigid) isotopy in $\mathbb{R}^3$. Moreover, there is an algorithm to decide, whether an element w of either RSG_n or NSG_n is central, which is linear in the length of w.

The encoding of the theorem is obtained by using 3-page embeddings of graphs into the product $\mathbb{Y} = T \times \mathbb{R}$, where T is the cone on 3 points.

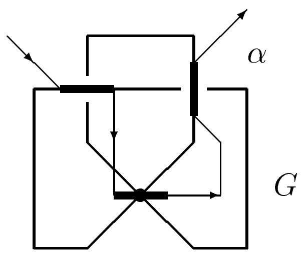

Fig. 1. Spatial graph G

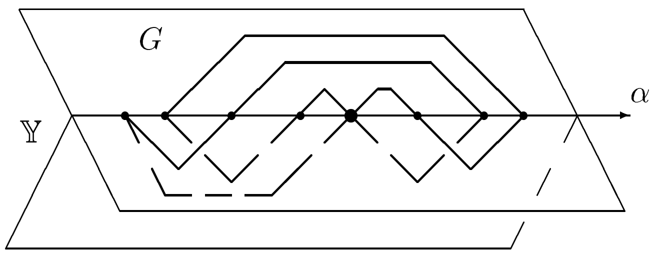

Fig. 2. A 3-page embedding of G, $w_G = a_0 a_1 b_2 d_1 x_{4,1} d_2 c_1 c_2$.

References

[1] *I. A. Dynnikov*, A new way to represent links. One-dimensional formalism and untangling technology, Acta Appl. Math., v. 69 (2001), p. 243–283.

[2] *V. A. Kurlin*, Three-page Dynnikov diagrams of spatial 3-valent graphs, Functional Analysis and Its Applications, v. 35 (2001), no. 3, p. 230–233.

[3] *V. A. Kurlin*, *A three-page presentation of spatial graphs*, submitted to Algebraic and Geometric Topology, see http://www.geocities.com/vak26