PRESERVATION OF CONVEXITY OF SOLUTIONS TO PARABOLIC EQUATIONS

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We consider the Cauchy problem for second order parabolic equations on $\mathbb{R}^n \times (0, T]$ of the form

$$\frac{\partial F}{\partial t} = \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t) \frac{\partial F}{\partial x_i} + c(x,t)F.$$

The problem under consideration is to find necessary and sufficient conditions on the operator that guarantee that the solutions to the equation remain convex, for each fixed time t, if the inital condition $F(x, 0)$ is a convex function.

In [1], we show that convexity is indeed preserved for solutions, given by the stochastic representation formula, to an equation of the form

$$F_t = a^2(x,t) F_{xx},$$

under very general conditions on $a(x, t)$.

In the present paper, we study the case of several spatial variables. In this case preservation of convexity is a rather rare property, in contrast to the case of one spatial dimension. Note that convexity is always preserved (for the solution of moderate growth) in the case of operators with constant coefficients since a solution is obtained by integrating the initial condition against a translation invariant positive kernel.

We give a necessary and sufficient condition for the infinitesimal preservation of convexity at some point. We call this condition LCP, an abbreviation for locally convexity preserving, and find a characterization of LCP in terms of a differential inequality on the coefficients of the operator. We then show that LCP holds if and only if convexity is preserved for solutions to the equation that are of polynomial growth.

We show the perhaps surprising result that for operators with bounded coefficients, it is only the operators with coefficients only depending on time that preserve convexity.

We apply the property of convexity preservation to study monotonicity properties of solutions to different parabolic equations and we also consider preservation of convexity for the Dirichlet problem on bounded domains in $\mathbb{R}^n$. Finally, we discuss extension of our results to nonlinear equations.

References

[1] Janson, S., & Tysk, J.: Volatility time and properties of option prices. *Ann. Appl. Probab.*(**13**)(2003) 890-913.