The Sobolev spaces and their applications to differential equations

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We have considered the conditions of solvability of the nonhomogeneous Dirichlet problem for non-linear differential equation of infinite order.

$$L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha}\Big(A_{\alpha}\big(x, \delta_k u(x)\big)\Big) = h(x),\ x \in \Omega \subset \mathbb{R}^n,\ n \geqslant 1, \quad (1)$$

$$D^{\omega}\ u(x)|_{\partial\Omega} = \Psi_{\omega}(x'),\ x' \in \partial\Omega,\ |\omega| = 0, 1, \dots . \quad (2)$$

The vector $\delta_k u(x)$ is

$$\{D_{\alpha} u\}_{|\alpha| \leqslant k} = \left\{ u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots, \frac{\partial^k u}{\partial x_n^k}, \right\}.$$

The "energy" space for the equation (1) is Sobolev space of infinite order

$$W^{\infty}\{a_{\alpha}, p\}_{(\Omega)} \equiv \left\{ u(x) \in C^{\infty}(\Omega):\ \rho(u) = \sum_{|\alpha|=0}^{\infty} a_{\alpha} \|D^{\alpha} u(x)\|_{L^p}^p < \infty \right\}. \quad (3)$$

The number $1 < p < \infty$ defines the growth of the functions $A_{\alpha}(x, \delta_k u(x))$. Considering boundary problem (1), (2), first of all it is necessary to study if a function satisfying the boundary conditions (2) might exist in the corresponding space of functions (3). These conditions were established in our works.

Studies of Sobolev spaces of infinite order and, in particular, conditions of the compact embedding

$$W^{\infty}\{a_{\alpha}, p\}_{(\Omega)} \subset W^{\infty}\{b_{\alpha}, p\}_{(\Omega)}$$

in the case of a bounded domain Ω have allowed to designate the superior out of two differential operators of infinite order. That allowed us to obtain the conditions of solvability of Dirichlet problem for non-linear differential equation of infinite order with subordinate terms. That theory allows us to obtain the solvability of the series problems, which could not be solved earlier. We demonstrate this by examples.