SYMPLECTIC 4-MANIFOLDS, SINGULAR PLANE CURVES AND ISOTOPY PROBLEMS

DENIS AUROUX

An important problem in 4-manifold topology is to understand which manifolds carry symplectic structures (i.e., closed non-degenerate 2-forms), and to develop invariants that can distinguish symplectic manifolds. Additionally, one would like to understand to what extent the category of symplectic manifolds is richer than that of Kahler (or complex projective) manifolds. Similar questions may be asked about singular curves inside, e.g., the complex projective plane. The two types of questions are related to each other via symplectic branched covers.

A branched cover of a symplectic 4-manifold with a (possibly singular) symplectic branch curve carries a natural symplectic structure. Conversely, using approximately holomorphic techniques it can be shown that every compact symplectic 4-manifold is a branched cover of the complex projective plane, with a branch curve presenting nodes (of both orientations) and complex cusps as its only singularities. The topology of the 4-manifold and that of the branch curve are closely related to each other; for example, using braid monodromy techniques to study the branch curve, one can reduce the classification of symplectic 4-manifolds to a (hard) question about factorizations in the braid group. Conversely, in some examples the topology of the branch curve complement (in particular its fundamental group) admits a simple description in terms of the total space of the covering.

In the language of branch curves, the failure of most symplectic manifolds to admit integrable complex structures translates into the failure of most symplectic branch curves to be isotopic to complex curves. While the symplectic isotopy problem has a negative answer for plane curves with cusp and node singularities, it is interesting to investigate this failure more precisely. Various partial results have been obtained recently about situations where isotopy holds (for smooth curves; for curves of low degree), and about isotopy up to stabilization or regular homotopy. On the other hand, many known examples of non-isotopic curves can be understood in terms of twisting along Lagrangian annuli (or equivalently, Luttinger surgery of the branched covers), leading to some intriguing open questions about the topology of symplectic 4-manifolds versus that of Kahler surfaces.