CLASSIFICATION OF INVARIANT MEASURES AND QUANTUM UNIQUE ERGODICITY

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In 1967 Furstenberg discovered a very surprising phenomenon: while both $Tx \to 2x \bmod 1$ and $sx \to 3x \bmod 1$ on $\mathbb{R}/\mathbb{Z}$ have many closed invariant sets, closed sets which are invariant under both T and S are very rare (indeed, are either finite sets of rationals or $\mathbb{R}/\mathbb{Z}$). Furstenberg also conjectured that a similar result holds for invariant measures. This conjecture is of course still open.

As has been shown by several authors, including Katok-Spatzier and Margulis, this phenomenon is not an isolated curiosity but rather a deep property of many natural $\mathbb{Z}^d$ and $\mathbb{R}^d$ actions ($d > 1$) with many applications.

Recently, there has been substantial progress in the study of measure rigidity of $\mathbb{R}^d$ and $\mathbb{Z}^d$-actions. In particular, in many cases of interest we have a full understanding of the invariant measures for which there is some element of the action which acts with positive entropy. This has previously been known only in the one-dimensional case and in some special $\mathbb{Z}^d$-actions by toral automorphisms.

While this is still far from resolving the issue, these partial results already powerful enough to prove results in other fields, especially number theory, which to date have been beyond the reach of more traditional techniques.

In particular, our (partial) understanding of a variant of the classical measure rigidity question enables us to prove a special but important case of Rudnick and Sarnak's Quantum Unique Ergodicity Conjecture.