NUMERICAL APPROXIMATIONS OF SINGULAR SOURCE TERMS IN DIFFERENTIAL EQUATIONS

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Singular source terms in differential equations appear in many different applications. Examples include multiphase flows, dendritic solidification, simulation of elastic boundaries in blood flow and subgrid wire modeling in computational electromagnetics. In the example of immiscible multiphase flow, the singular source term arises from the surface tension forces that are acting on the moving and deforming interfaces that are separating any two different liquids.

Such singular terms in differential equations pose severe challenges for numerical approximations on regular grids, and regularization of the singularities is a very useful technique for their representation on the grid.

Let $\Gamma \subset \mathbb{R}^d$ be a $d-1$ dimensional continuous and bounded surface and assume that that we have a Dirac delta function of variable strength with support on Γ. We want to replace the Dirac delta function δ with support on Γ by a more regular function δ_ε, which can be used on standard computational grids in connection to numerical solution of differential equations with singular source terms and quadrature with singular integrands. We are in particular interested in multi-dimensional singular functions.

The analysis of the errors associated with regularization can for a wider support of the regularization be performed by dividing the error into an analytical and a numerical part, analyzing each part separately. This leads to certain continuous moment conditions together with requirements on the regularity of the approximation to achieve a certain accuracy. There will also be an optimal scaling of the width of the support of the regularization as a function of the grid size.

However, for the practically preferred case of narrow support of the regularizations, over a few grid cells only, discrete effects will be important, and the total error must be analyzed directly. In one dimension, the analysis now instead leads to certain discrete moment conditions to be imposed on the delta function approximation, to ensure a certain accuracy.

We discuss two different approaches to extend this one-dimensional approximation to several dimensions. The first is what we call the product rule, which we show to naturally carry over the properties and accuracy obtained in one dimension to several dimensions. The second approach is the seemingly natural way to define the multi-dimensional delta function approximation by using the normal distance to Γ as the argument for the one-dimensional approximation. We however show that this is an inconsistent approximation that may lead to $O(1)$ errors.

We will also to some extent discuss the less singular case of discontinuous coefficients.