Quantitative stability in geometric & functional inequalities

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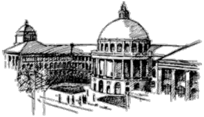
Göran Gustafsson lectures



Stability problems

Geometric and functional inequalities play a crucial role in several problems arising in the calculus of variations, partial differential equations, geometry, physics, etc.

Classical examples:

- **Isoperimetric inequalities;**
- **Sobolev inequalities;**
- **Gagliardo-Nirenberg inequalities;**
- **Brunn-Minkowski.**

Basic question:

(1) Find the sharp constant for these inequalities.

(2) Characterize the minimizers.

These are by now well understood.

What next?

(3) Are minimizers stable? That is, if a function/set almost attains the equality in one of the previous inequalities, is it close to one of the minimizers?

Overview of the talk

1. Stability for isoperimetric inequalities
2. Stability for the Brunn-Minkowski inequality
3. Stability for the Sobolev inequality: minimizers vs critical points

Part 1: Isoperimetric inequalities

Isoperimetric inequalities and stability

Classical isoperimetric inequality: For any bounded open smooth set $E \subset \mathbb{R}^n$, the perimeter $P(E)$ controls the volume $|E|$:

$$P(E) \geq n|B_1|^{1/n}|E|^{(n-1)/n}.$$

Moreover equality holds if and only if E is a ball.

Stability question: If E is "almost a minimizer", does this imply that E is close to a ball, if possible in some quantitative way?

Isoperimetric deficit of E:

$$\delta(E) := \frac{P(E)}{n|B_1|^{1/n}|E|^{(n-1)/n}} - 1\,.$$

- $\delta(E) \geq 0$.
- $\delta(E) = 0 \quad \Leftrightarrow \quad E$ is a ball.

Asymmetry index of E:

$$A(E) := \inf_x \left\{ \frac{|E\Delta(B_r(x))|}{|E|} : |B_r| = |E| \right\}$$

Here $E\Delta F$ denotes the symmetric difference between the sets E and F, i.e., $E\Delta F := (E \setminus F) \cup (F \setminus E)$.

Question: can we find positive constants $C = C(n)$ and $\alpha = \alpha(n)$ such that

$$A(E) \leq C\,\delta(E)^\alpha\,?$$

Remark: by testing the above inequality on a sequence of ellipsoids converging to B_1, we get $\alpha \leq 1/2$.

This is actually the sharp result:

Theorem (Fusco-Maggi-Pratelli, 2008)

The stability result holds with $\alpha = 1/2$.

The proof of Fusco-Maggi-Pratelli uses symmetrization techniques which are very specific to the Euclidean case.

We want to describe a different approach which has the advantage to work for much more general perimeter-type functionals. More precisely, we replace the classical perimeter by

$$P_f(E) := \int_{\partial E} f(\nu_E)$$

with f positively 1-homogeneous and convex, and we look for the stability of the corresponding isoperimetric inequality (the so-called **Wulff inequality**).

Theorem (Figalli-Maggi-Pratelli, 2010)

The stability result holds for P_f with $\alpha = 1/2$.

Also, C is an explicit constant depending only on the dimension (and not on f).

Application

Consider a liquid drop/crystal $E \subset \mathbb{R}^n$ under the action of an exterior potential $g : \mathbb{R}^n \to \mathbb{R}$.

Then E minimizes the free energy under a volume constraint:

$$\text{free energy} = \text{surface energy} + \text{bulk energy}$$

More precisely, E minimizes

$$F \mapsto P_f(F) + \int_F g(x)\,dx$$

among all sets F with $|F| = |E|$.

Let $K = K_f$ be the minimizer when $g \equiv 0$ with $|K| = 1$.

Theorem (Figalli-Maggi, 2011)

Let $m := |E|$, $\hat{E} := m^{-1/n}E$ *(so that* $|\hat{E}| = 1$*).*

Then, up to a translation, if $m \ll 1$*:*

1. $\|\partial\hat{E} - \partial K\|_{L^\infty} \lesssim m^{1/n}$;
2. *If* f *smooth and "uniformly convex", then* $\|\partial\hat{E} - \partial K\|_{C^2} \lesssim m^{2/[n(n+2)]}$;
3. *If* $n = 2$ *and* f *is crystalline (equivalently,* K *is a convex polygon), then* $\hat{E}$ *is a convex polygon with sides parallel to that of* K*.*

Theorem (Figalli-Zhang, 2021)

Last statement true also for $n \geq 3$.

Gromov's proof of the isoperimetric inequality

Given E smooth and bounded, consider the probability densities

$$f(x) := \frac{\chi_E(x)}{|E|}, \qquad g(y) := \frac{\chi_{B_1}(y)}{|B_1|}.$$

By Optimal Transportation Theory, there exists $\varphi : \mathbb{R}^n \to \mathbb{R}$ convex such that $T := \nabla\varphi$ sends f onto g, that is

$$T_\# f = g \qquad \Leftrightarrow \qquad \int_A g = \int_{T^{-1}(A)} f \quad \forall\, A.$$

Properties of T:

1. $|T| \leq 1$ in E (since $T(E) \subset B_1$)
2. $\det(DT) = |B_1|/|E|$ (since $T_\# f = g$)
3. $\operatorname{div} T \geq n\big(\det(DT)\big)^{1/n}$ (wait for the next slide).

Then:

$$P(E) = \int_{\partial E} 1 \overset{(1)}{\geq} \int_{\partial E} |T| \geq \int_{\partial E} T \cdot \nu_E$$

$$= \int_E \operatorname{div} T \overset{(3)}{\geq} n \int_E (\det(DT))^{1/n} \overset{(2)}{=} n|B_1|^{1/n}|E|^{(n-1)/n}.$$

Proof of (3)

Since $T = \nabla\varphi$ with φ convex, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $D^2\varphi$ are non-negative.

Hence:

$$\operatorname{div} T = \Delta\varphi = n\left(\frac{1}{n}\sum_{i=1}^{n}\lambda_i\right) \geq n\left(\prod_{i=1}^{n}\lambda_i\right)^{1/n} = n\big(\det(DT)\big)^{1/n},$$

where we used the arithmetic-geometric inequality.

Strengths of this proof:

- It works also for the functional P_f.

- It is very robust.

In particular, by carefully making “quantitative” each inequality one can prove the desired stability result.

Part 2: The Brunn-Minkowski inequality

Semisum of sets

Let $A \subset R^n$ be a Borel set with $|A| > 0$. We define

$$\frac{A+A}{2} := \left\{ \frac{a+a'}{2} \ : \ a, a' \in A \right\}.$$

Obviously $\frac{A+A}{2} \supset A$, hence

$$\left| \frac{A+A}{2} \right| \geq |A|.$$

In addition equality holds iff

$$|\mathrm{co}(A) \setminus A| = 0,$$

where $\mathrm{co}(A)$ denote the convex hull of A.

This is a particular case of a more general inequality: the *Brunn-Minkowski inequality*.

Given $A, B \subset \mathbb{R}^n$ Borel, with $|A|, B| > 0$, define

$$\frac{A+B}{2} := \left\{ \frac{a+b}{2} \, : \, a \in A, \, b \in B \right\}.$$

Then

$$\left| \frac{A+B}{2} \right|^{1/n} \geq \frac{|A|^{1/n} + |B|^{1/n}}{2}.$$

In addition, equality holds iff A and B are homothetic convex sets: there exist $\alpha, \beta > 0$, $v, w \in \mathbb{R}^n$, K convex, such that

$$A \subset \alpha K + v, \qquad |(\alpha K + v) \setminus A| = 0,$$

$$B \subset \beta K + w, \qquad |(\beta K + w) \setminus B| = 0.$$

The stability issue

Question: are these results stable?

That is, if for instance

$$\left|\frac{A+A}{2}\right| = |A| + \varepsilon$$

with $\varepsilon \ll |A|$, is it true that A is close to its convex hull?

Stability for BM on convex sets

Let A, B be bounded convex set with $0 < \lambda \leq |A|, |B| \leq \Lambda$.

Define

$$\delta := \left|\frac{A+B}{2}\right|^{1/n} - \frac{|A|^{1/n} + |B|^{1/n}}{2}.$$

Then $\delta \geq 0$, and we would like to show that δ controls the distance between A and B.

First of all, renormalize A so that it has the same measure of B: if $\gamma := \frac{|B|^{1/n}}{|A|^{1/n}}$ then

$$|\gamma A| = |B|.$$

Define a “distance” between A and B:

$$d(A, B) := \min_{x \in \mathbb{R}^n} |B \Delta (x + \gamma A)|,$$

where

$$E \Delta F := (E \setminus F) \cup (F \setminus E).$$

Theorem (Figalli-Maggi-Pratelli, 2009)

There exists $C = C(n, \lambda, \Lambda)$ such that

$$d(A, B) \leq C\sqrt{\delta}.$$

Remark: the exponent $1/2$ is optimal and C is explicit.

Stability when A=B, $n = 1$

Without loss of generality $|A| = 1$. Define

$$\delta(A) := |A + A| - 2|A|.$$

REMARK: $\delta(A)$ cannot control in general $|\mathrm{co}(A) \setminus A|$:

One may have $\delta(A) = 1 (= |A|)$ and $|\mathrm{co}(A) \setminus A|$ arbitrarily large.

By Freiman, this is the only thing that can go wrong:

Theorem (Freiman, 1959)

Let $A \subset \mathbb{R}$ be a Borel set with $|A| = 1$, and denote by $\mathrm{co}(A)$ its convex hull. If $\delta(A) < 1$ then

$$|\mathrm{co}(A) \setminus A| \leq \delta(A).$$

To be precise, Freiman's Theorem is about the structure of additive subsets of $\mathbb{Z}$.

Stability when A=B, $n \geq 1$

Let $A \subset \mathbb{R}^n$ with $|A| = 1$, and define

$$\delta(A) := \left|\frac{A+A}{2}\right| - |A|.$$

Theorem (Christ, 2012)

If $\delta(A) \to 0$ *then*

$$|\mathrm{co}(A) \setminus A| \to 0.$$

Theorem (Figalli-Jerison, 2015)

Let $n \geq 1$. There exist computable dimensional constants $\delta_n, C_n > 0$ such that if $\delta(A) \leq \delta_n$, then

$$|\mathrm{co}(A) \setminus A| \leq C_n \delta(A)^{\alpha_n}, \qquad \alpha_n := \frac{1}{8^{n-1} n! \left[(n-1)!\right]^2}.$$

Many improvement and generalizations in the last 6 years...

Part 3: The Sobolev inequality

The Sobolev inequality

Let $n \geq 3$.

The Sobolev inequality states

$$(Sob_2) \qquad \|\nabla u\|_{L^2} \geq S\|u\|_{L^{2^*}}$$

where

$$2^* = \frac{2n}{n-2}.$$

Also, equality holds iff

$$u = c\, U_{\lambda,z}, \qquad U_{\lambda,z}(x) = [n(n-2)]^{\frac{n-2}{4}} \frac{\lambda^{\frac{n-2}{2}}}{(1+\lambda|x-z|^2)^{\frac{n-2}{2}}}.$$

Stability question

Assume

$$\|\nabla u\|_{L^2} = S^{n/2} = \|\nabla U_{\lambda,z}\|_{L^2}$$

and

$$\delta(u) := \|\nabla u\|_{L^2} - S\|u\|_{L^{2^*}} \ll 1.$$

Is u close to some $U_{\lambda,z}$?

A sharp stability result

Theorem (Bianchi-Egnell, 1991)

Let $\|\nabla u\|_{L^2} = S^{n/2}$. *Then*

$$\min_{\lambda,z} \|\nabla u - \nabla U_{\lambda,z}\|_{L^2} \leq C(n)\,\delta(u)^{1/2}.$$

Remark: The power $1/2$ is optimal.

Critical points

Let u be a minimizer of (Sob_2). Then

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Big(\|\nabla u + \varepsilon \nabla \varphi\|_{L^2} - S\|u + \varepsilon\varphi\|_{L^{2^*}} \Big)$$

for all $\varphi \in W^{1,2}$.

From this, we find the Euler-Lagrange equation

$$-\Delta u = \gamma u^p, \qquad p = 2^* - 1,$$

for some $\gamma > 0$.

Characterization of critical points

Up to rescaling u, one can assume $\gamma = 1$.

Theorem (Obata, 1972; Gidas-Ni-Nirenberg, 1979)

$$-\Delta u = u^p \qquad \textit{in } \mathbb{R}^n \quad \Longrightarrow \quad u = U_{\lambda,z} \quad \textit{for some } \lambda, z.$$

Stability question: Assume

$$-\Delta u = u^p + R, \qquad R \approx 0.$$

Is u close to some $U_{\lambda,z}$?

Issue: What is the natural space for R?

$$u \in W^{1,2} \cap L^{2^*} \quad \Rightarrow \quad \Delta u + u^p \in (W^{1,2})' = W^{-1,2}.$$

This suggests the definition

$$\delta(u) := \|\Delta u + u^p\|_{W^{-1,2}}.$$

In this setting, *bubbling* can occur.

Consider $u = \sum_{i=1}^m U_{\lambda_i,z_i}$, where the functions $\{U_{\lambda_i,z_i}\}$ are supported far away from each other. Then

$$-\Delta u = -\sum_{i=1}^m \Delta U_{\lambda_i,z_i} = \sum_{i=1}^m U_{\lambda_i,z_i}^p \approx \Big(\sum_{i=1}^m U_{\lambda_i,z_i}\Big)^p = u^p.$$

A qualitative stability result

Theorem (Struwe, 1984)

Assume
$$\int |\nabla u|^2 \leq \left(L + \frac{1}{2}\right) S^n.$$

For all $\varepsilon > 0$ *there exists* $\delta_\varepsilon > 0$ *such that*

$$\delta(u) \leq \delta_\varepsilon \quad \Rightarrow \quad \left\| \nabla u - \sum_{i=1}^{m} \nabla U_{\lambda_i, z_i} \right\|_{L^2} \leq \varepsilon$$

for some $m \leq L$.

Also
$$\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} \geq \frac{1}{\varepsilon}, \qquad \lambda_i \lambda_j |z_i - z_j|^2 \geq \frac{1}{\varepsilon}.$$

Theorem (Ciraolo-Figalli-Maggi, 2017; Figalli-Glaudo, 2020)

Assume

$$\int |\nabla u|^2 \leq L + \frac{1}{2} S^n.$$

Then:

$$L = 1 \quad \Rightarrow \quad \|\nabla u - \nabla U_{\lambda,z}\|_{L^2} \leq C(n)\,\delta(u);$$

$$L > 1,\, 3 \leq n \leq 5 \quad \Rightarrow \quad \left\| \nabla u - \sum_{i=1}^{m} \nabla U_{\lambda_i,z_i} \right\|_{L^2} \leq C(n)\,\delta(u).$$

Remark: power is 1 instead of $1/2$. Also, $n \leq 5$ is optimal.

This result allows us to prove rates of convergence to equilibrium for the Yamabe flow on the sphere.

Thanks for your attention!