STABILITY FOR B-M

$$\frac{A+B}{2} := \left\{ \frac{a+b}{2} \;\middle|\; a\in A,\ b\in B \right\}$$

(BM) $$\left|\frac{A+B}{2}\right|^{\frac1n} \geq \frac{|A|^{\frac1n} + |B|^{\frac1n}}{2}$$

Also, $=$ holds iff

$A \subseteq \alpha K + v$, $|(\alpha K + v)\setminus A| = 0$

$B \subseteq \beta K + w$, $|(\beta K + w)\setminus B| = 0$

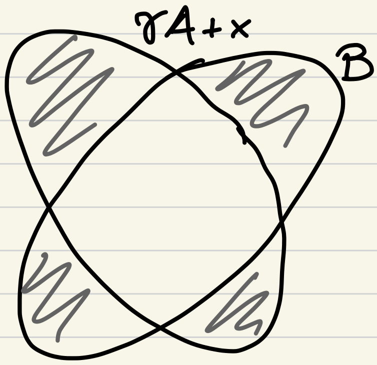
Define

$$\delta := \left|\frac{A+B}{2}\right|^{\frac12} - \frac{|A|^{\frac1n} + |B|^{\frac1n}}{2} \geq 0$$

Assume $\lambda \leq |A|, |B| \leq \Lambda$

Set $\gamma := \frac{|B|^{\frac1n}}{|A|^{\frac1n}}$, s.t. $|\gamma A| = |B|$.

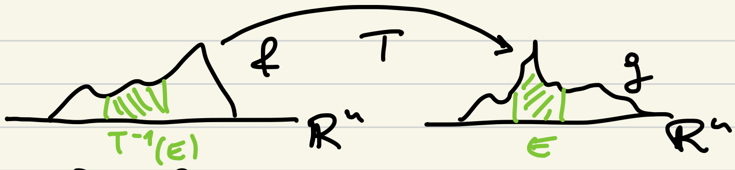
$$d(A,B) = \inf_x |(\gamma A + x)\,\Delta\, B|$$



THM If A, B are convex, then

$$d(A,B) \leq C(n,\lambda,\Lambda)\sqrt{\delta}.$$

OPTIMAL TRANSPORT



$\int f = \int g = 1$

Let $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given,

$c(x,y)$ = cost to move a unit of mass from x to y

We say that T sends f to g if

$$T_\# f = g \iff \int_E g = \int_{T^{-1}(E)} f \quad \forall E$$

$$\mathrm{Cost}(T) := \int_{\mathbb{R}^n} c(x, T(x))\, f(x)\, dx$$

Rmk If T is a diffeo and $T_\# f = g$,

$$\int_{T^{-1}(E)} f(x)\,dx = \int_E g(y)\,dy \overset{y=T(x)}{=} \int_{T^{-1}(E)} g(T(x))\, |\det \nabla T(x)|\, dx$$

$$\Downarrow$$

$$\boxed{f(x) = g(T(x))\, |\det \nabla T(x)| \quad \text{a.e.}}$$

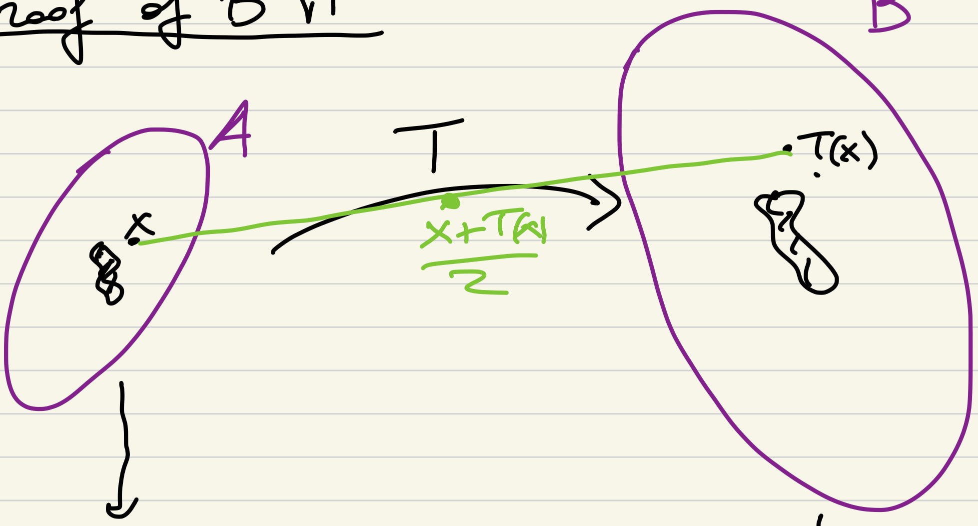
THM (Brenier 1987)

If $c(x,y) = |x-y|^2$ then there exists a unique optimal transport map, i.e., $\exists !\ T$ s.t.

$$\mathrm{Cost}(T) = \min_{S_\# f = g} \mathrm{Cost}(S)$$

In addition, $T = \nabla \varphi$ for some $\varphi : \mathbb{R}^n \to \mathbb{R}$ convex.

Proof of BM



$$f = \frac{\chi_A}{|A|} \qquad\qquad g = \frac{\chi_B}{|B|}$$

$$\chi_A(x) = \begin{cases} 1 & \text{in } A \\ 0 & \text{outside} \end{cases}$$

By Brenier, $\exists!$ optimal T, $T = \nabla\varphi$, φ convex

PROPERTIES

① $T(A) = B$

$\gamma = \frac{|B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}}$

② $|\det \nabla T| = \frac{f(x)}{g(T(x))} = \frac{1}{|A|} \cdot \left(\frac{1}{|B|}\right)^{-1} = \frac{|B|}{|A|} = \gamma^n.$

Define

$$\left(\frac{Id+T}{2}\right)(A) := \left\{ \frac{x+T(x)}{2} : x \in A \right\} \subseteq \frac{A+B}{2}$$

$$\left|\frac{A+B}{2}\right| \geqslant \left|\left(\frac{Id+T}{2}\right)(A)\right| = \int_A \left|\det\left(\nabla \frac{Id+T}{2}\right)\right|$$

$$= \frac{1}{2^n} \int_A |\det(Id + \nabla T)|$$

$$T = \nabla\varphi \to \nabla T = \nabla^2\varphi \qquad \nabla^2\varphi = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_i \geq 0$

$$|\det(Id+\nabla T)| = |\det(Id+\nabla^2\varphi)| = \begin{pmatrix} 1+\lambda_1 & & 0 \\ & \ddots & \\ 0 & & 1+\lambda_n \end{pmatrix}$$

$$= \prod_{i=1}^{n}(1+\lambda_i)$$

$$\geq \left[1+\left(\prod_{i=1}^{n}\lambda_i\right)^{\frac{1}{n}}\right]^n$$

$$= \left[1+(\det\nabla T)^{\frac{1}{n}}\right]^n$$

$$= (1+\gamma)^n$$

$$\Downarrow$$

$$\left|\frac{A+B}{2}\right| \geq \frac{1}{2^n}\int_A (1+\gamma)^n = \frac{(1+\gamma)^n}{2^n}|A|$$

$$= \left(\frac{|A|^{\frac{1}{n}}+|B|^{\frac{1}{n}}}{2}\right)^n.$$

$$\gamma = \frac{|B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}}}$$

$$\delta \gtrsim \int_A \underbrace{\Big[|\det(Id+\nabla T)| - (1+\gamma)^n\Big]}_{\gtrsim \frac{|\nabla T-\gamma Id|^2}{|\nabla T|}} dx$$

$$\Downarrow$$

$$\sqrt{\delta} \gtrsim \int_A |\underbrace{\nabla T-\gamma Id}_{\nabla(T(x)-\gamma x)}| \rightsquigarrow |T(x)-\gamma x| \lesssim \sqrt{\delta}$$

$$\Downarrow$$

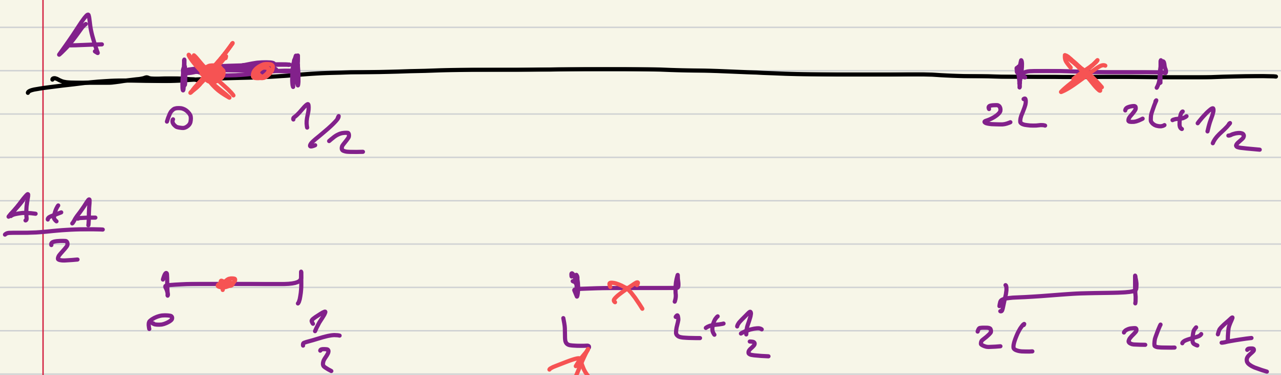
$$|\underbrace{T(A)}_{B} - \gamma A| \lesssim \sqrt{\delta}$$

Rmk

If $A=B \Rightarrow T(x)=x \leftarrow$ No info

Non Convex World

$A=B$, $n=1$ $\qquad \delta(A)=|A+A|-2|A|$



$$\frac{\delta(A)}{2} = \left|\frac{A+A}{2}\right| - |A| = \frac{1}{2} \Rightarrow \delta(A)=1=|A|$$

$$|co(A)\setminus A| = 2L - \frac{1}{2} \gg 1.$$

THM (Freiman)

$|A|=1$. Assume $\delta(A)<1$. Then

$$|co(A)\setminus A| \le \delta(A).$$

Pf of Freiman

$|A| = \# A$

LEMMA 1

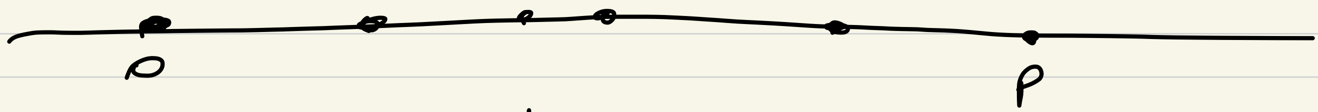
$\emptyset \neq A, B \subseteq \mathbb{Z}/p\mathbb{Z}$, p prime

$$\Downarrow$$

$$\#(A+B) \geq \min\{\#A + \#B - 1, p\}$$

Pf By induction on $\#B$.

Lemma 2 $A \subseteq \mathbb{Z}$, $\min A = 0$, $\max A = p$, p prime



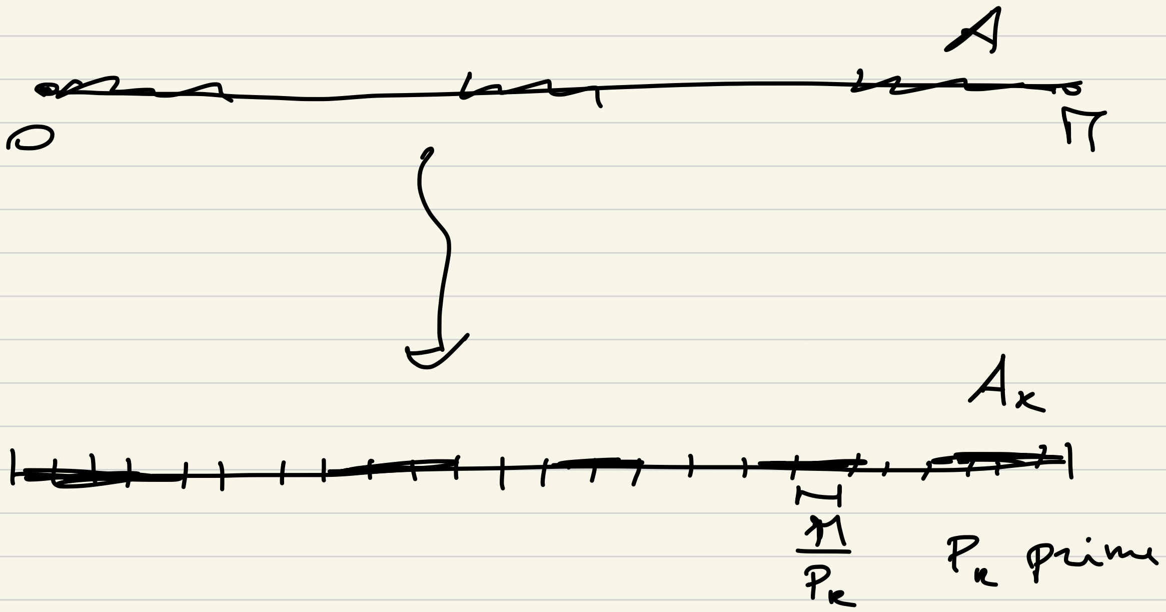
Assume $|A+A| < 3|A| - 3$. Then

$$\underbrace{|\{0, \dots, p\} \setminus A|}_{co(A) \setminus A} \leq \underbrace{|A+A| - 2|A|}_{\delta(A)} + 1.$$

L1 ⇒ L2

Pf Freiman in $\mathbb{Z}$

Let $A \subseteq \mathbb{R}$, $co(A) = [0, M]$.



$\square$

THM (2019 - 2020)

Let $A \subseteq \mathbb{R}^n$, $\delta(A) = \left|\frac{A+A}{2}\right| - |A|$

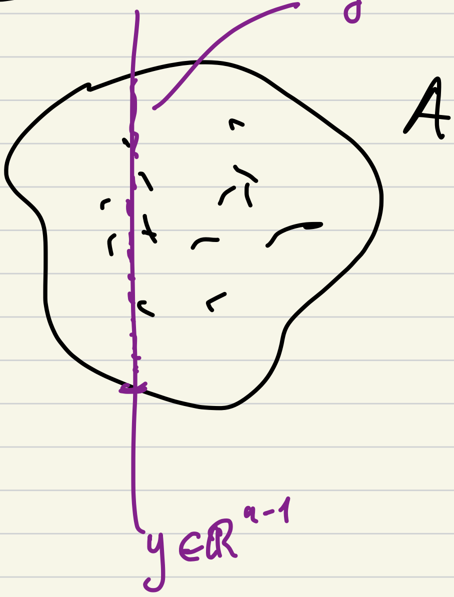
$\delta(A) << 1$. Then

$$|co(A) \setminus A| \leq C\,\delta(A)$$

THM (2017)

Stability with $C\,\delta(A,B)^{\alpha}$.

IDEA

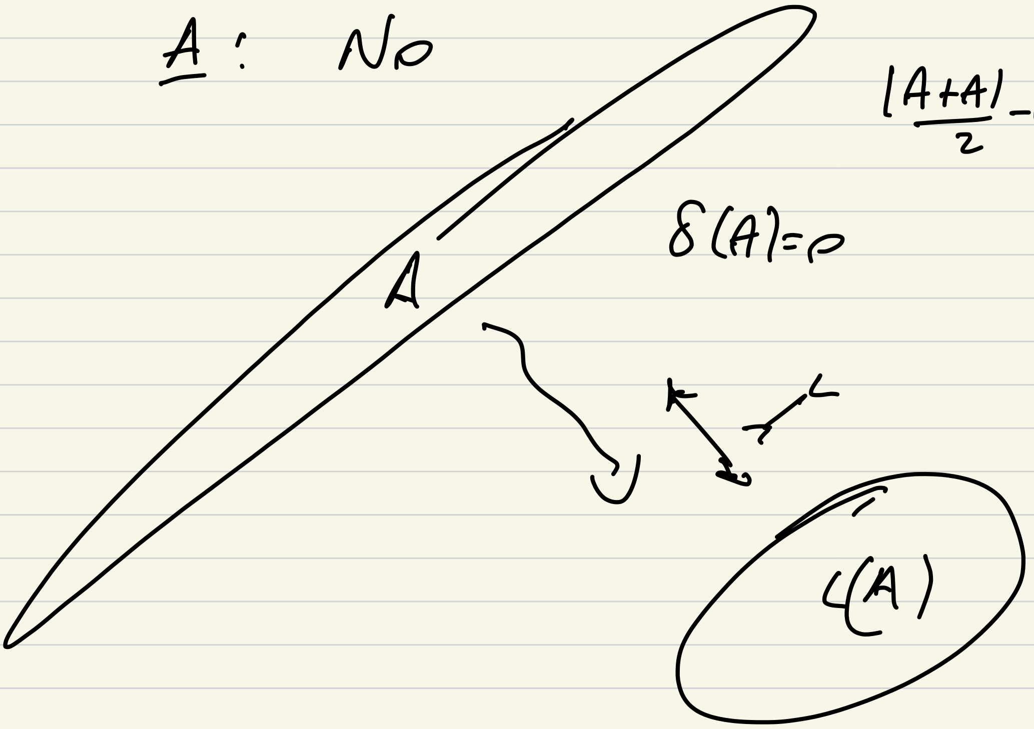


$$\delta(A) \ll 1$$
$$\Downarrow$$
$$\delta(A_y) \ll 1 \text{ for many } y.$$
$$\Downarrow \text{ 1D}$$
$$A_y \sim \text{interval}$$

$$\boxed{\underline{Q}: \quad \delta(A) \ll 1 \overset{?}{\Longrightarrow} A \text{ bdd}}$$

$\underline{A}$: No

$$\frac{|A+A|}{2} - |A|$$



STABILITY SOBOLEV

SOBOLEV : $1 < p < n$

$$\|\nabla u\|_{L^p(\mathbb{R}^n)} \geq S \, \|u\|_{L^{p^*}(\mathbb{R}^n)} \quad \forall u \in C_c^1$$

$$S = S(p,n) > 0, \qquad p^* = \frac{np}{n-p}$$

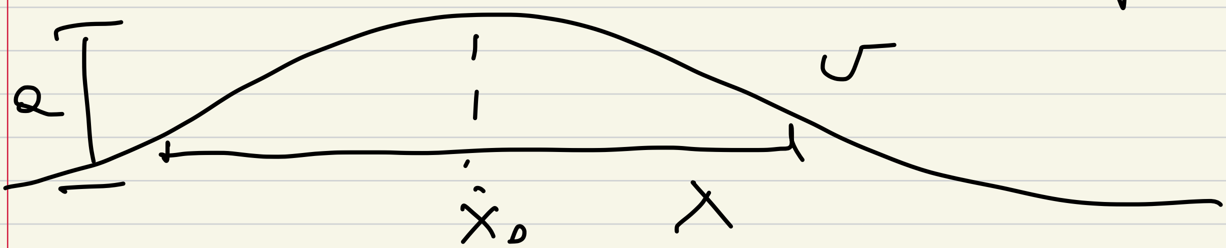
$$\text{Equality} \iff u = v_{a,\lambda,x_0}$$

$$v_{a,\lambda,x_0}(x) = \frac{a}{\left(1 + \lambda\,|x-x_0|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}}$$

$$a \in \mathbb{R}, \quad \lambda > 0, \quad x_0 \in \mathbb{R}^n$$

$$\mathcal{M} = \left\{ v_{a,\lambda,x_0} \;\middle|\; \begin{matrix} a \in \mathbb{R}, \ \lambda > 0, \\ x_0 \in \mathbb{R}^n \end{matrix} \right\}$$

$(n+2)$-dim manifold



DEF $\quad \delta(u) := \dfrac{\|\nabla u\|_{L^p}}{\|u\|_{L^{p^*}}} - S$

PROP $\quad \delta \geq 0, \quad \{\delta = 0\} = \mathcal{M}$

Q : If $\delta(u) \ll 1$, is u close to $\mathcal{M}$?

A : Yes.

Q (Brezis-Lieb, 1985) Can we find $C, \alpha > 0$ s.t.

$$\delta(u) \geq C \operatorname{dist}(u, \mathcal{M})^{\alpha} \, ?$$

to be defined

THM (Bianchi-Egnell, 1991)

P=2 $$\delta(u) \geq c \inf_{v \in \mathcal{M}} \left(\frac{\|\nabla u - \nabla v\|_{L^2}}{\|\nabla u\|_{L^2}} \right)^2$$

RmK
- Optimal in the power 2
- $c > 0$ is NOT computable

2 Extensions : $\begin{cases} p \neq 2 & (1) \\ \text{critical pts} & (2) \end{cases}$

$\boxed{p \neq 2}$ • Cianchi - Fusco - Maggi - Pratelli 2009

$$\delta(u) \geq c \inf_{v} \left(\frac{\| u - v \|_{L^{p^*}}}{\| u \|_{L^{p^*}}} \right)^{\alpha}$$

$\alpha = \alpha(n, p)$

• F - Neumayer 2019, $p > 2$

$$\delta(u) \geq c \inf_{v} \left(\frac{\| \nabla u - \nabla v \|_{L^p}}{\| \nabla u \|_{L^p}} \right)^{\alpha}$$

Neumayer 2020, $p < 2$

<u>THM</u> (F - Zhang, preprint)

$$\boxed{\delta(u) \geq c \inf_{v} \left(\frac{\| \nabla u - \nabla v \|_{L^p}}{\| \nabla u \|_{L^p}} \right)^{\alpha}}$$

$\alpha = \max\{2, p\}$.

<u>OPTIMALITY of α</u>

• Fix $v \in \mathcal{M}$, define $u_k(x) = v(A_k x)$

$$A_k = \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \\ & & 1 & \\ 0 & & & 1 + \frac{1}{k} \end{pmatrix}$$

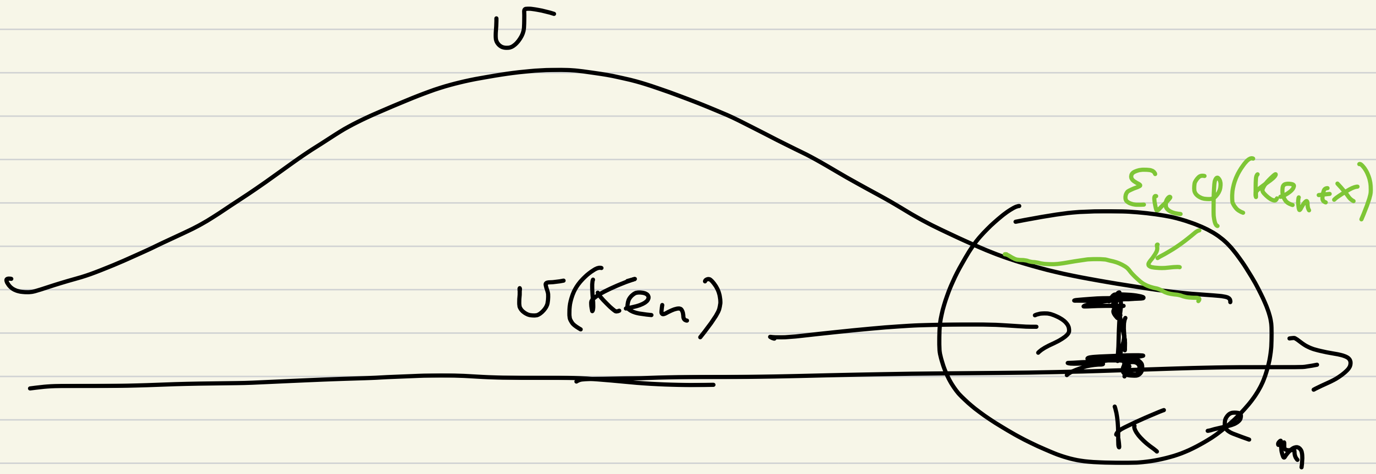
$\delta(u_k) \sim k^{-2}$, $\| \nabla u_k - \nabla v \|_{L^p} \sim k^{-1}$

$\Longrightarrow \alpha \geq 2$

• Fix $v \in \mathcal{M}$,

$$u_k(x) = v(x) + \varepsilon_k \varphi(Ke_n + x)$$

$$\varphi \in C_c^\infty, \quad 0 \le \varphi \le 1$$



$$v(Ke_n) << \varepsilon_k << 1$$

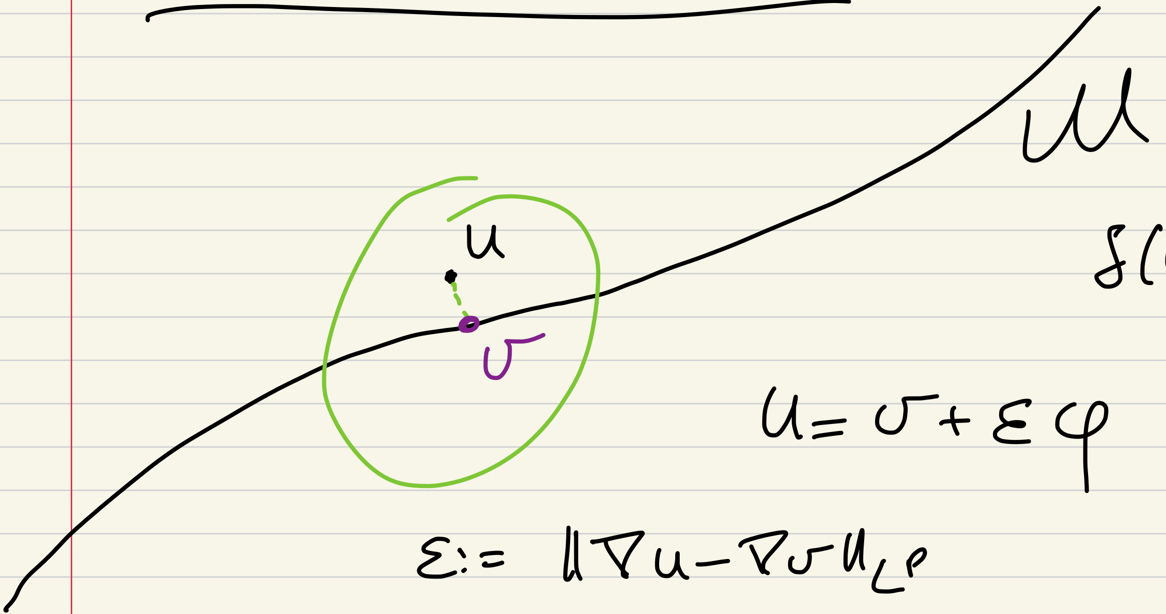
$$\|\nabla u_k\|_{L^p}^p \simeq \int |\nabla u_k|^p \simeq \int |\nabla v|^p + \varepsilon_k^p \int |\nabla \varphi|^p$$

$$\|u_k\|_{L^{p^*}}^{p^*} \simeq \int v^{p^*} + \varepsilon_k^{p^*} \int \varphi^{p^*}$$

$$\int(u_k) \sim \varepsilon_k^p \qquad \|\nabla u_k - \nabla v\|_{L^p} \sim \varepsilon_k$$

$$\Longrightarrow \quad \alpha \ge p.$$

IDEA OF PF



$\delta(u)$

$$u = v + \varepsilon\varphi$$

$$\varepsilon := \|\nabla u - \nabla v\|_{L^p}$$

$$\varphi := \frac{u-v}{\varepsilon}.$$

v s.t. $\inf \dfrac{\|\nabla u - \nabla v\|_{L^p}}{\|\nabla u\|_{L^p}}$ attained

$$\delta(u) = \delta(v+\varepsilon\varphi)$$
$$= \varepsilon^2 Q_v[\varphi] + o(\varepsilon^2\|\nabla\varphi\|_{L^p}^2)$$

KEY FACT (B-E '91)

$$Q_v[\varphi] \gtrsim \|\nabla\varphi\|_{L^2}^2$$

$$\delta(u) \geq \varepsilon^2\|\nabla\varphi\|_{L^2}^2 = \|\nabla u - \nabla v\|_{L^2}^2$$

$\underline{p>2}$

$$\delta(u) = \delta(v+\varepsilon\varphi) = \varepsilon^2 \underbrace{Q_v[\varphi]}_{\sim} + o\left(\boxed{\varepsilon^2 \|\nabla\varphi\|^2_{L^p}}\right)$$

$W^{1,p}$

$$\int |\nabla v|^{p-2} |\nabla\varphi|^2$$

$W^{1,2}$-norm

$$\varepsilon^2 \int |\nabla v|^{p-2} |\nabla\varphi|^2 + (p-2)\, |w(x)|^{p-2} \left(\frac{|\nabla u| - |\nabla v|}{\varepsilon}\right)^2$$

$w = w(\nabla u(x), \nabla v(x))$

CRITICAL PT

$v \in \mathcal{M}$
$v > 0$

$$\rightsquigarrow \underbrace{\frac{d}{d\varepsilon}\left(\|\nabla v + \varepsilon\nabla\varphi\|_{L^p} - S\, \|v+\varepsilon\varphi\|_{L^{p^*}}\right)}_{\substack{\shortparallel \\ 0}}$$

$$-\Delta_p v = c\, v^{p^*-1}$$

up to replace v by αv, we can assume $c = 1$

$p=2$

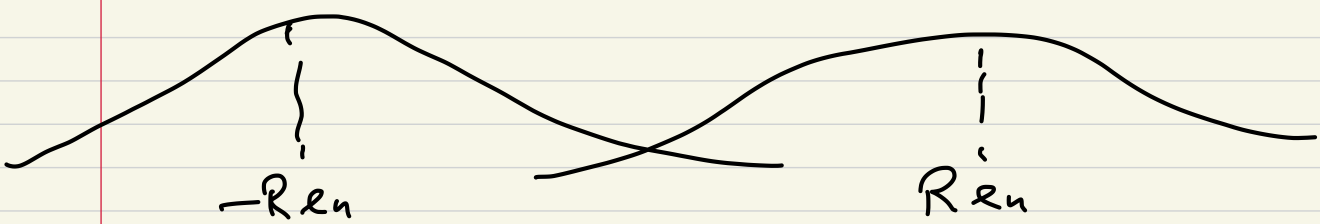
DEF $\delta(u) := \|\Delta u + u^{2^*-1}\|_*$
$u>0$

THM $\delta(u)=0,\ u>0 \Rightarrow u \in \mathcal{M}.$

Q: If $\delta(u) \ll 1 \Rightarrow u$ close to $\mathcal{M}$?

A: No

$$u_R(x) = v(x+Re_n) + v(x-Re_n)$$

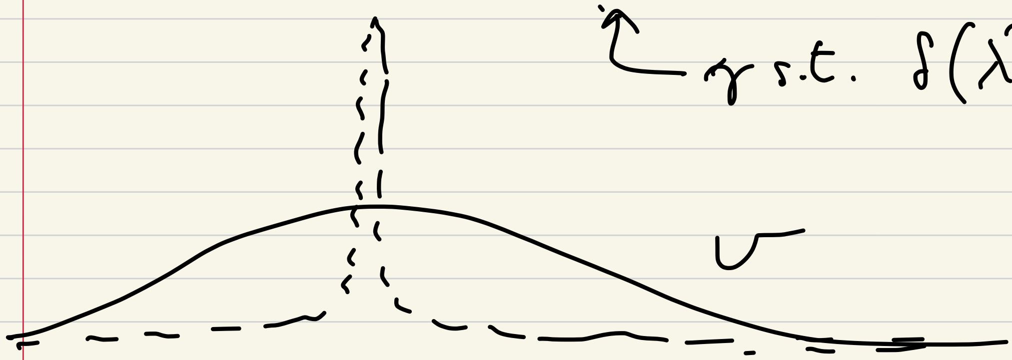


$$\|\Delta u_R + \underbrace{u_R^{2^*-1}}_{\approx}\| \longrightarrow 0$$

$$v(x+Re_n)^{2^*-1} + v(x-Re_n)^{2^*-1}$$

$$u_{\cancel{R}}(x) = v(x) + \lambda^\gamma v(\lambda x)$$

γ s.t. $\delta(\lambda^\gamma v(\lambda x)) = 0$



THM (Struwe 1984)

$n \geq 3$, $\nu \geq 1$, $\{u_k\} \subseteq W^{1,2}(\mathbb{R}^n)$

$$\begin{cases} \left(\nu - \frac{1}{2}\right) S^n \leq \int |\nabla u_k|^2 \leq \left(\nu + \frac{1}{2}\right) S^n \\ \delta(u_k) \longrightarrow 0 \end{cases}$$

$$\Downarrow$$

$\exists \{\lambda_k^i, x_k^i\}_{i=1}^{\nu}$ s.t.

$$\left\| \nabla u_k - \sum_{i=1}^{\nu} \nabla \sigma_{1,\lambda_k^i, x_k^i} \right\|_{L^2} \longrightarrow 0$$

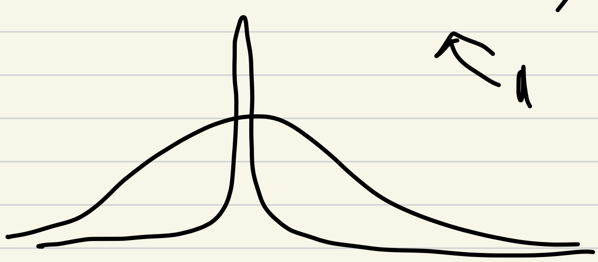
$$\Delta\left(\sigma + \overbrace{\lambda^{\gamma}\sigma(\lambda\cdot)}^{w}\right) + \left(\sigma + \overbrace{\lambda^{\gamma}\sigma(\lambda\cdot)}^{w}\right)^{2^*-1}$$

$$= \Delta\sigma + \Delta w + \underbrace{(\sigma + w)^{2^*-1}}_{= \sigma^2 + w^2 + 2\sigma w}$$

$$\frac{2^* - 1 = 2}{\boxed{n=6}}$$

$$= 2\sigma w = 2\sigma\lambda^{\gamma}\sigma(\lambda\cdot) \xrightarrow[\lambda\to\infty]{} 0$$

1 $\quad \|_{\circledast}$



$$\delta(u) = \|\Delta u + u^{2^*-1}\|_{\bigcirc} \qquad u \in W^{1,2}$$

$$u \in W^{1,2} \hookrightarrow \Delta u \in W^{-1,2}$$

<u>Connection to isop ineq</u>

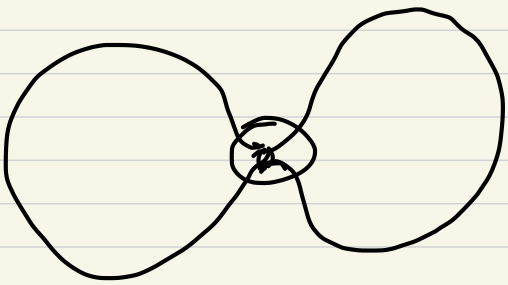
E optimal in isop ineq $\Rightarrow H_{\partial E} \equiv \text{const}$

<u>THM</u> (Alexandrov)

∂E smooth embedded closed surface,

$H_{\partial E} \equiv \text{const} \Rightarrow \partial E$ sphere

<u>Q</u> $H_{\partial E} \approx \text{const} \Rightarrow \partial E \approx \text{sphere}$?



No

THM (Ciraolo - F - Maggi 2017, F-Glaudo 2020, Deng - Sun - Wei 2021)

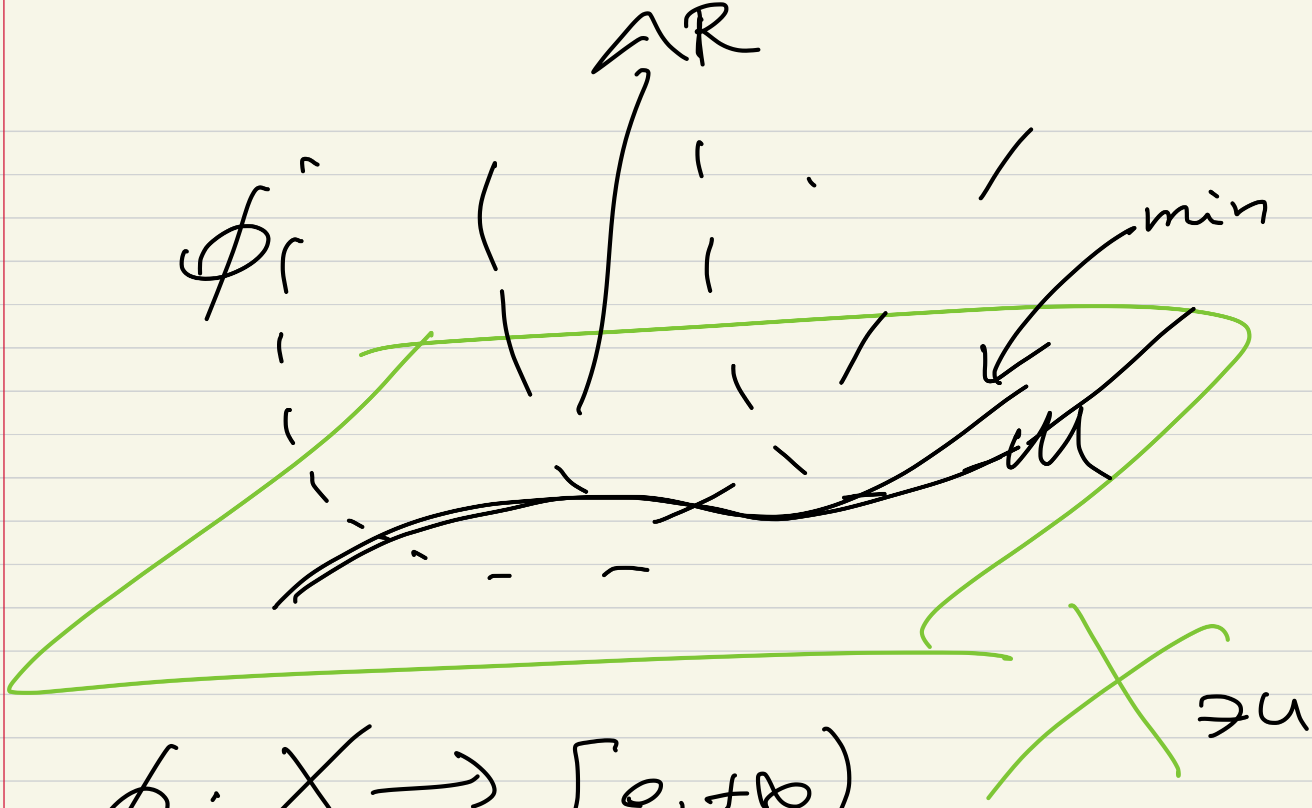
- $\nu = 1 \Longrightarrow \|\nabla u - \nabla \sigma\|_{L^2} \leq C\,\delta(u)^{\overset{\alpha}{\uparrow}}_{1}$
- $\nu > 1 \Longrightarrow$ " " if $3 \leq n \leq 5$
- $\nu > 1,\ n \geq 6 \Longrightarrow$

$$\|\nabla u - \sum_i \nabla \sigma_i\|_{L^2} \leq \begin{cases} C\,\delta(u)\sqrt{|\log \delta(u)|} & n = 6 \\ C\,\delta(u)^{\frac{n+2}{2(n-2)}} & n \geq 7 \end{cases}$$

OPTIMAL

$\forall\ \sigma_1, \sigma_2 \in \mathcal{M},\ \delta(\sigma_i) = 0$

$$\|\sigma_1 \sigma_2^{2^*-2}\|_{(L^{2^*})'} \approx \int \sigma_1 \sigma_2^{2^*-1} \quad \text{if } n \leq 5$$

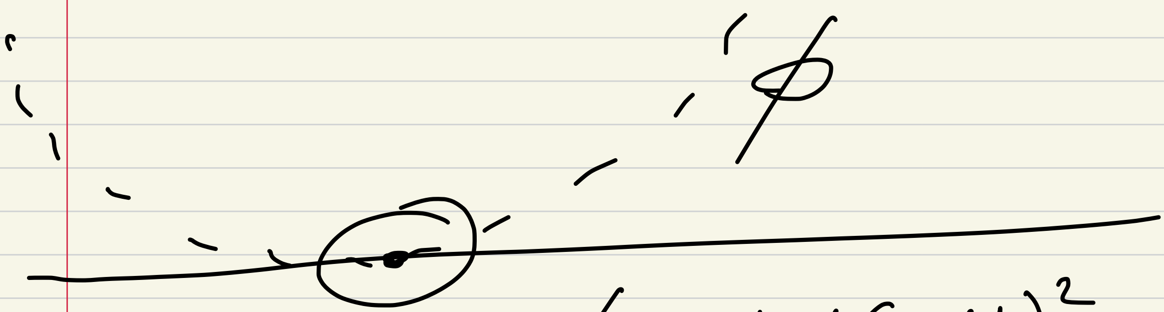


$$\phi : X \longrightarrow [0, +\infty)$$

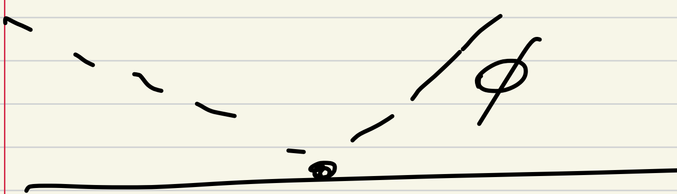
$$\{\phi = 0\} = \mathcal{M}$$

$$\delta(u) = \phi(u)$$

$$\phi(u) \gtrsim \operatorname{dist}(u, \mathcal{M})^{\alpha}$$



$$\phi \sim \operatorname{dist}(\,, \mathcal{M})^2$$



$$\phi \sim \operatorname{dist}(\cdot, \mathcal{M})$$