1. Introduction

This note reports on a recent work of Cabré and Capella [CC2] concerning a special class of radial solutions of semilinear elliptic equations. It is the class of semi-stable solutions, which includes local minimizers, minimal solutions, extremal solutions, and also certain solutions found between a sub and a supersolution.

Consider the problem

\[
\begin{cases}
  -\Delta u = g(u) & \text{in } \Omega \\
  u \geq 0 & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1)

where \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain, and \( g : [0, +\infty) \to \mathbb{R} \) is locally Lipschitz. Given a weak solution (a notion described below, in next section) \( u \) of (1), consider the quadratic form

\[
Q_u(\varphi) := \int_\Omega |\nabla \varphi|^2 - g'(u)\varphi^2,
\]

(2)

with \( \varphi \) a \( C^\infty \) function with compact support in \( \Omega \). Note that \( Q_u \) corresponds to the second variation of the energy associated to (1).

We say that \( u \) is semi-stable if \( Q_u(\varphi) \geq 0 \) for all such \( \varphi \). If \( u \) is bounded, this is equivalent to the nonnegativeness of the first eigenvalue in \( \Omega \) of the linearized problem \(-\Delta = g'(u)\) of (1) at \( u \).

In [CC2] we establish sharp pointwise, \( L^q \), and \( W^{k,q} \) estimates for semi-stable radial solutions. One of our results is the following (below, in Theorem 5 we state further estimates).

**Theorem 1 ([CC2]).** Let \( g : [0, +\infty) \to \mathbb{R} \) be locally Lipschitz and \( u \in H^1_0(B_1) \) be a semi-stable radially decreasing weak solution of (1) with \( \Omega = B_1 \subset \mathbb{R}^n \) (the unit ball).

If \( n \leq 9 \), then \( u \in L^\infty(B_1) \). In addition, if \( g \geq 0 \) and \( n \leq 9 \), then \( \|u\|_{L^\infty(B_1)} \leq C_n\|u\|_{L^1(B_1)} \) for some constant \( C_n \) depending only on \( n \).

The theorem holds for every locally Lipschitz nonlinearity, and hence the result goes beyond the usual regularity given by the Sobolev critical exponent. What is used here to gain regularity is the semi-stability property of the weak solution \( u \)

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(together with the assumption that \( u \) is radial and \( u \in H^1_0 \)). The theorem does not hold for unstable solutions. In addition, the condition \( n \leq 9 \) is optimal. A similar result in the nonradial case is still open. See next section and [CC2] for more details on all these comments.

The precise semi-stability hypothesis of Theorem \([\text{I}]\) is that \( Q_u(\varphi) \geq 0 \) for all \( C^\infty \) function \( \varphi \) with compact support in \( B_1 \setminus \{0\} \). Since a priori \( u \) could be unbounded at the origin, this guarantees that \( Q_u(\varphi) \) is well defined.

The class of semi-stable solutions of (1) in general \( \Omega \) contains the following three interesting types of solutions (see [CC2] for more details):

- **Local minimizers of the energy.** A local minimizer is a function that minimizes the energy in a \( C^1_0(\Omega) \) neighborhood around it (i.e., a minimizer under every small enough \( C^1 \) perturbation vanishing on \( \partial \Omega \)). In the radial case, to include weak solutions possibly unbounded at the origin, we consider local minimizers under small perturbations with compact support in \( B_1 \setminus \{0\} \); see Definition 1.2 and Theorem 1.3 of [CC2] for more details on this.

- **Absolute minimizers between a subsolution and a supersolution.** Given a subsolution \( u \) and a supersolution \( \bar{u} \) of (1) with \( u < \bar{u} \), there exists at least one absolute minimizer \( u \) of the energy in the convex set of functions lying between \( u \) and \( \bar{u} \). By the strong maximum principle, such function \( u \) is a solution of (1).

- **Minimal and extremal solutions.** As we will see, this is indeed a subclass of the previous class formed by absolute minimizers between a subsolution and a supersolution. Minimal and extremal solutions motivated our work and we describe them next (see [C] for a more detailed survey on this topic). We put emphasis on the regularity of the extremal solution, both in general domains and in the radial case. As we will see below, [CC2] gives optimal regularity results for radial extremal solutions.

### 2. MINIMAL AND EXTREMAL SOLUTIONS

Consider the semilinear elliptic problem

\[
\begin{align*}
-\Delta u &= \lambda f(u) & & \text{in } \Omega \\
u &= 0 & & \text{on } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a smooth bounded domain, \( n \geq 2, \lambda \geq 0, \) and the nonlinearity \( f : [0, +\infty) \to \mathbb{R} \) satisfies

\[ f \text{ is } C^1, \text{ increasing and convex, } f(0) > 0, \text{ and } \lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \]
Since $f > 0$, we consider nonnegative solutions of $(3_\lambda)$. The cases $f(u) = e^u$ and $f(u) = (1 + u)^p$, with $p > 1$, are classical examples of such nonlinearities.

It is well known that there exists a parameter $\lambda^*$ with $0 < \lambda^* < \infty$, called the extremal parameter, such that if $0 \leq \lambda < \lambda^*$ then $(3_\lambda)$ has a minimal classical solution $u_\lambda$. On the other hand, if $\lambda > \lambda^*$ then $(3_\lambda)$ has no classical solution. Here, classical means bounded, while minimal means smallest.

The set $\{u_\lambda : 0 \leq \lambda < \lambda^*\}$ forms a branch of solutions increasing in $\lambda$. Moreover, every solution $u_\lambda$ is stable, in the sense that the first Dirichlet eigenvalue of the linearized problem at $u_\lambda$ is positive:

$$\mu_1\{-\Delta - \lambda f'(u_\lambda) ; \Omega\} > 0.$$ 

In particular, for the quadratic form associated to the linearized problem we have $Q_{u_\lambda}(\varphi) \geq 0$ for all $\varphi \in H^1_0(\Omega)$. Recall that $Q_{u_\lambda}$ is given by (2) with $g'(u)$ replaced here by $\lambda f'(u_\lambda)$. This condition, that above we have called semi-stability, is equivalent to $\mu_1 \geq 0$, where $\mu_1$ is the first eigenvalue of the linearized problem as above.

The existence of the branch $\{u_\lambda : 0 \leq \lambda < \lambda^*\}$ can be proved using the Implicit Function Theorem (starting from $\lambda = 0$). The solution $u_\lambda$ may also be obtained by the monotone iteration procedure, with $0 < \lambda < \lambda^*$ fixed, starting from $u \equiv 0$ (note that $u \equiv 0$ is a strict subsolution of the problem). This is the reason why $u_\lambda$ is the minimal solution, i.e., it is smaller than every other solution or supersolution of $(3_\lambda)$.

As a consequence, $u_\lambda$ must coincide with the absolute minimizer of the energy in the closed convex set of functions lying between 0 (a strict subsolution) and $u_\lambda$ (since there is no other solution smaller than $u_\lambda$). In particular, by the discussion in Section 1 on this type of minimizers, $u_\lambda$ must a semi-stable solution—something that we already knew, but that here is seen as a consequence of being the minimal or smallest solution.

The increasing limit of $u_\lambda$ as $\lambda \uparrow \lambda^*$ is called the extremal solution $u^*$. It is proved in [BCMR] that $u^*$ is a weak solution of $(3_\lambda)$ for $\lambda = \lambda^*$ in the following sense. We say that $u$ is a weak solution of $(3_\lambda)$ if $u \in L^1(\Omega)$, $f(u)\delta \in L^1(\Omega)$, and

$$-\int_\Omega u \Delta \zeta = \int_\Omega \lambda f(u) \zeta$$

for all $\zeta \in C^2(\Omega)$ with $\zeta = 0$ on $\partial \Omega$, where $\delta(x) = \text{dist}(x, \partial \Omega)$ denotes the distance to the boundary of $\Omega$.

The extremal solution $u^*$ is semi-stable (this simply follows by monotone convergence from the semi-stability of $u_\lambda$ for $\lambda < \lambda^*$). $u^*$ may be classical or singular depending on each problem. For instance, when $f(u) = e^u$ we have:

**Theorem 2 (CR, MR, JL).** Let $u^*$ be the extremal solution of $(3_\lambda)$.

(i) If $f(u) = e^u$ and $n \leq 9$, then $u^* \in L^\infty(\Omega)$ (i.e., $u^*$ is classical in every $\Omega$).

(ii) If $\Omega = B_1$, $f(u) = e^u$, and $n \geq 10$, then $u^* = \log(1/|x|^2)$ and $\lambda^* = 2(n-2)$.

There is an analogous result for $f(u) = (1 + u)^p$. In this case, the explicit radial solution is given by $|x|^{-2/(p-1)} - 1$ and coincides with the extremal solution $u^*$ in $\Omega = B_1$ for some values of $n$ and $p$ (see [BV, CC2] for more details).
Part (i) of Theorem 2 was proven by Crandall and Rabinowitz [CR] and by Mignot and Puel [MP]. The proof uses the semi-stability of the minimal solutions, as follows. One takes $\varphi = e^{\alpha u} - 1$ in the semi-stability condition $Q_{u_\lambda}(\varphi) \geq 0$. One then uses that $u_\lambda$ is a solution of the problem. This leads to an $L^1(\Omega)$ bound for $e^{n/2}u_\lambda$, uniform in $\lambda$, for some exponents $n$. That is, $-\Delta u_\lambda$ is uniformly bounded in $W^{2,\tilde{n}}(\Omega)$. If $n \leq 9$, the exponent $n/2$ turns out to be bigger than $n/2$ and, therefore, we have a uniform $L^\infty(\Omega)$ bound for $u_\lambda$.

Part (ii) of the theorem had been proved by Joseph and Lundgren [JL] in their exhaustive study of the radial case. More recently, Brezis and Vázquez [BV] have introduced a simple approach to this question, based on the following characterization of singular extremal solutions by their semi-stability property:

**Theorem 3 (BV).** Let $u \in H^1_0(\Omega), u \notin L^\infty(\Omega)$, be an unbounded semi-stable weak solution of $(3_\lambda)$ for some $\lambda > 0$. Then, $\lambda = \lambda^*$ and $u = u^*$.

The idea behind the theorem is that, for each $\lambda > 0$, $(3_\lambda)$ has at most one semi-stable solution—a consequence of the convexity of $f$.

Following [BV], we can deduce part (ii) of Theorem 2 from Theorem 3. Indeed, let $\Omega = B_1$ and $\bar{u} = \log(1/|x|^2)$. A direct computation shows that this function is a solution of $(3_\lambda)$ for $\lambda = \lambda^* = 2(n - 2)$. The linearized operator at $\bar{u}$ is given by

$$L\varphi = -\Delta \varphi - 2(n - 2)e^{\pi \varphi} = -\Delta \varphi - \frac{2(n - 2)}{|x|^2} \varphi.$$

If $n \geq 10$ then the first eigenvalue of $L$ in $B_1$ satisfies $\mu_1\{L; B_1\} \geq 0$. This is a consequence of Hardy’s inequality:

$$\frac{(n - 2)^2}{4} \int_{B_1} \frac{\varphi^2}{|x|^2} \leq \int_{B_1} |\nabla \varphi|^2 \quad \forall \varphi \in H^1_0(B_1),$$

and the fact that $(n - 2)^2/4 \geq 2(n - 2)$ if $n \geq 10$. Applying Theorem 3 to $(\pi, \lambda)$ we deduce part (ii) of Theorem 2.

**Theorem 2** is a precise result on the boundedness of $u^*$ when $f(u) = e^u$. However, to establish regularity of $u^*$ for a general $f$ satisfying (4) —a question raised by Brezis and Vázquez [BV]— is a much harder task. The best known results in general domains and for general nonlinearities satisfying (4) are due to Nedev [N1, N2]:

**Theorem 4 (N1, N2).** Let $u^*$ be the extremal solution of $(3_\lambda)$.

(i) If $n \leq 3$, then $u^* \in L^\infty(\Omega)$ (for every $\Omega$).

(ii) If $n \leq 5$, then $u^* \in H^1_0(\Omega)$ (for every $\Omega$).

(iii) For every dimension $n$, if $\Omega$ is strictly convex then $u^* \in H^1_0(\Omega)$.

The proof of this theorem uses a very refined version of the method of [CR, MP] described above in relation with Theorem 2.

There are still many questions to be answered in the case of general $\Omega$ and $f$. For instance, it is not known if an extremal solution may be singular in dimensions $4 \leq n \leq 9$, for some domain and nonlinearity.
However, the radial case $\Omega = B_1$ has been recently settled by the author and Capella [CC2]. The result establishes optimal regularity results for general $f$. To state it, we define exponents $q_k$ for $k \in \{0, 1, 2, 3\}$ by

$$\begin{cases} 
\frac{1}{q_k} = \frac{1}{2} - \frac{\sqrt{n-1}}{n} + \frac{k-2}{n} & \text{for } n \geq 10 \\
q_k = +\infty & \text{for } n \leq 9.
\end{cases}$$

(5)

Note that all the exponents are well defined and satisfy $2 < q_k \leq +\infty$. In the same way as Theorem 4, the following result holds for every $f$ satisfying (4):

**Theorem 5** ([CC2]). Let $\Omega = B_1$ and $u^*$ be the extremal solution of (3λ).

(i) If $n \leq 9$, then $u^* \in L^\infty(B_1)$.

(ii) If $n = 10$, then $u^*(|x|) \leq C \log(1/|x|)$ in $B_1$ for some constant $C$.

(iii) If $n \geq 11$, then

$$u^*(|x|) \leq C |x|^{-n/2+\sqrt{n-1}+2} \sqrt{\log(1/|x|)} \quad \text{in } B_1$$

(6)

for some constant $C$. In particular, $u^* \in L^q(B_1)$ for every $q < q_0$. Moreover, for every $n \geq 11$ there exists $p_n > 1$ such that $u^* \notin L^{p_n}(B_1)$ when $f(u) = (1 + u)^{p_n}$.

(iv) $u^* \in W^{k,q}(B_1)$ for every $k \in \{1, 2, 3\}$ and $q < q_k$. In particular, $u^* \in H^3(B_1)$ for every $n$. Moreover, for all $n \geq 10$ and $k \in \{1, 2, 3\}$,

$$|\partial^k u^*(|x|)| \leq C |x|^{-n/2+\sqrt{n-1}+2-k} (1 + \sqrt{\log(1/|x|)}) \quad \text{in } B_1$$

for some constant $C$. Here $\partial^k u^*$ denotes the $k$-th derivative of the radial function $u^*$.

Theorem 2, which deals with $f(u) = e^u$, shows the optimality of (i) and (ii) in Theorem 5 including the logarithmic pointwise bound of part (ii). The $L^q$ regularity stated in part (iii) ($q < q_0$) is also optimal. This is shown considering $f(u) = (1 + u)^{p_n}$ (for an explicit $p_n$), in which case $u^*(|x|) = |x|^{-n/2+\sqrt{n-1}+2} - 1$. This function differs from the pointwise power bound (6) for the factor $\sqrt{\log(1/|x|)}$. It is an open problem to know if this logarithmic factor in (6) can be removed. The exponents $q_k$ in the Sobolev estimates of part (iv) are optimal. This follows immediately from the optimality of $q_0$ and the fact that all $q_k$ are related by optimal Sobolev embeddings.

The proof of Theorem 5 was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^n$ for $n \leq 7$. The key idea is to take $\varphi = (r^{-\beta} - 1) u_\alpha$, where $r = |x|$, and compute $Q_{u^*}((r^{-\beta} - 1) u_\alpha)$ in the semi-stability property satisfied by $u^*$. The nonnegativeness of $Q_{u^*}$ leads to an $L^2$ bound for $u_\alpha r^{-\alpha}$, with $\alpha$ depending on the dimension $n$. This is the key point in the proof. A similar method was employed in [CCT] to study stability properties of radial solutions in all of $\mathbb{R}^n$.

Theorem 5 has been extended in [CCS] to equations involving the $p$-Laplacian and having general semilinear reaction terms. Previously, Boccardo, Escobedo and Peral [BEP] had extended Theorem 2 on the exponential nonlinearity to the $p$-Laplacian case in general domains.
REFERENCES


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