Best constants in some exponential Sobolev inequalities

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Abstract

A Pohozaev identity is used to classify the radial solutions of a quasi-linear equation with exponential nonlinearity. The results are applied to find the infimum of the non-local functional

$$\mathcal{F}(\lambda, u) = \frac{1}{n} \int_{\Omega} |\nabla u|^n dx - \lambda F\left(\int_{\Omega} e^u dx\right), \quad u \in W_0^{1,n}(\Omega),$$

for various nonlinearities $F$, where $\Omega$ is a bounded domain of $\mathbb{R}^n$ and $\lambda$ a real parameter. Our results generalize the case when $F(s) = \log s$.

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1 Introduction

Given a bounded domain $\Omega$ of $\mathbb{R}^n$, consider the Sobolev space $W_0^{1,n}(\Omega)$ defined as the completion with respect to the norm $\|\nabla u\|_n = \left(\int_{\Omega} |\nabla u|^n\right)^{\frac{1}{n}}$ of the class of smooth functions in $\Omega$ having compact support. It is well-known that this Sobolev space embeds in all spaces $L^q(\Omega)$ with $q \in [1, \infty)$. The independent works of Judović [11], Pohozaev [20] and Trudinger [27] have extended the classical
Sobolev inequalities by proving the existence of constants $\mu, C_n > 0$ depending only on the dimension $n$ such that
\[
\int_{\Omega} e^{\mu \left( \frac{|u|}{\|\nabla u\|_n} \right)^{\frac{n}{n-1}}} \leq C_n, \quad \forall u \in W^{1,n}_0(\Omega) \setminus \{0\},
\] (1.1)
where $f_\Omega f := \frac{1}{|\Omega|} \int_{\Omega} f \, dx$. A sharp form of this inequality has been obtained later by Moser [18], who showed that (1.1) holds if and only if $\mu \leq \mu_n := n \omega_1^{\frac{1}{n-1}}$ where $\omega$ denotes the measure of the unit sphere $S^{n-1} \subset \mathbb{R}^n$.

Inequality (1.1), commonly called Moser-Trudinger inequality, implies the following estimate (as indicated in Section 5):
\[
\int_{\Omega} e^{u} \leq e^{A_n} e^{\alpha_n \|\nabla u\|^n_n}, \quad \forall u \in W^{1,n}_0(\Omega),
\] (1.2)
where $A_n \geq 0$ is some universal constant and $\alpha_n$ is given by
\[
\alpha_n = (n - 1)^{n-1} n^{-2} \omega^{-1}.
\] (1.3)

In two dimensions, this inequality has revealed itself very useful in statistical physics ([5], [14], [1]) and is also related to the geometrical problem of prescribing Gauss curvature (see [18]). For this dimension the critical value (1.3) is given by $\alpha_2 = \frac{1}{16 \pi}$.

Similarly to what has been done by Talenti [25] for the Sobolev inequalities $\|\nabla u\|_p \leq C \|u\|_p^r$ with $p < n$, one may ask for the best constants $C_n$ and $A_n$ in the inequalities (1.1) and (1.2). By using Schwarz symmetrization it is enough to treat this question when $\Omega$ is a ball. For this domain, Carleson and Chang [6] proved the existence of an extremal function for the Moser-Trudinger inequality (1.1) and that the best constant $C_n$ is strictly greater than $1 + e^{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}$.

Concerning the related inequality (1.2) it was already noted in [5] that for a ball in the plane, there are no functions realizing equality in (1.2). Furthermore Carleson and Chang [6] found that $A_n = \sum_{k=1}^{n-1} \frac{1}{k}$ is the optimal constant in (1.2) when the domain is a ball in $\mathbb{R}^n$. In other words any $u \in W^{1,n}_0(\Omega)$ satisfies the inequality
\[
\frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - (n \alpha_n)^{-1} \log \left( \int_{\Omega} e^n \, dx \right) > -(n \alpha_n)^{-1} \sum_{k=1}^{n-1} \frac{1}{k},
\] (1.4)
and the constant on the right hand-side of (1.4) is optimal when the domain is a ball.
Similar inequalities hold on compact manifolds. For example on the two-dimensional canonical sphere, it was proved by Onofri [19] (see also [10]) that any \( u \in W^{1,2}(S^2) \) of average zero satisfies

\[
\frac{1}{2} \int_{S^2} |\nabla u|^2 - 8\pi \log \left( \int_{S^2} e^u \right) \geq 0. \tag{1.5}
\]

Using a stereographical transformation, Beckner [4] proved that such an inequality is equivalent to the following one on a disk \( B \) of \( \mathbb{R}^2 \)

\[
\frac{1}{2} \int_B |\nabla u|^2 - 8\pi \left\{ \log \left( \int_B e^u \right) + \left( \int_B e^u \right)^{-1} \right\} \geq -8\pi, \tag{1.6}
\]

in the space of non-negative functions of \( W^{1,2}_0(B) \). Motivated by this equivalence, Kim [15] has studied inequality (1.6) in higher dimension and proved that

\[
\frac{1}{n} \int_{\Omega} |\nabla u|^n - (n\alpha_n)^{-1} \left\{ \log \left( \int_{\Omega} e^u \right) + \left( \int_{\Omega} e^u \right)^{-1} \right\} \geq -(n\alpha_n)^{-1} \sum_{k=1}^{n-1} \frac{1}{k}, \tag{1.7}
\]

for any \( u \in W^{1,n}_0(\Omega) \) with \( u \geq 0 \).

Note that the constants appearing on the right hand-side of (1.7) and (1.4) are exactly the same. This is not a just a simple coincidence, and the aim of the present paper is to exhibit a general setting in which both these sharp inequalities are covered. More precisely given a function \( F : (0, \infty) \to \mathbb{R} \), and \( \lambda \in (0, \infty) \) we shall consider the general non-local functional

\[
\mathcal{F}(\lambda, u) := \frac{1}{n} \int_{\Omega} |\nabla u|^n dx - \lambda F \left( \int_{\Omega} e^u \right), \quad u \in W^{1,n}_0(\Omega). \tag{1.8}
\]

Under the assumptions

\[
F \in C^1(0, \infty), \quad \lim_{s \to \infty} \left\{ F(s) - \log s \right\} = 0, \quad F' \geq 0, \tag{1.9}
\]

inequality (1.2) easily implies that the functional (1.8) is well-defined, admits a minimizer when \( \lambda < (n\alpha_n)^{-1} \), is bounded from below at \( \lambda = (n\alpha_n)^{-1} \), while beyond that value appropriate test functions show that \( \mathcal{F}(\lambda, \cdot) \) is unbounded from below. Further any critical point of \( \mathcal{F}(\lambda, \cdot) \) satisfies in the weak sense the non-local quasilinear equation

\[
-\Delta_n u = \frac{\lambda}{|\Omega|} f \left( \int_{\Omega} e^u \right) e^u, \quad u \in W^{1,n}_0(\Omega), \tag{1.10}
\]

where \( \Delta_n u := \text{div} (|\nabla u|^{n-2} \nabla u) \) denotes the \( n \)-Laplacian and \( f := F' \).
Thanks to the assumption $f \geq 0$ we may assume one of the minimizer to be non-negative and therefore by using Schwarz symmetrization, we are reduced to consider Problem (1.10) in the class of radially symmetric functions define on a ball of same volume as $\Omega$. This restriction on the volume is actually irrelevant since both the functional $F(\lambda, \cdot)$ and Problem (1.10) are invariant under scaling. The study of (1.10) in the class of radial functions on a ball will be based on the generalized Pohozaev identity as established through the works of Pohozaev [20], Pucci and Serrin [21], Degiovanni et al. [8]. Since this integral identity will play a central role in all our arguments, we will give an independent proof in Section 2. This identity has mainly be used to prove non-existence of solutions in several nonlinear problems. In the present paper we apply it to classify the radial solutions of the equations $-\Delta_n v = \pm e^v$ and of its non-local variant (1.10). Our first main result stated in a simpler form is the following:

**Theorem 1.1.** Let $\Omega$ be ball of $\mathbb{R}^n$. Then Problem (1.10) admits a positive radial solution if and only if there exists $\sigma > 1$ satisfying

$$\frac{n\alpha_n \lambda f(\sigma)}{\sigma} = \frac{1}{\sigma} \left| 1 - \frac{1}{\sigma} \right|^{n-1}. \quad (1.11)$$

A similar result holds for the existence of negative radial solutions and we refer to Theorem 4.1 for a complete statement. By applying our results to the special case where $f(s) = \frac{1}{s} \left| 1 - \frac{1}{s} \right|^{k-1} \left( 1 - \frac{1}{s} \right)$, $k \geq 1$, we generalize and complete the results obtained previously in [15] for $k = 2$ (see our Proposition 4.2). In particular inequality (1.6) can be extended as follows:

**Theorem 1.2.** Assume $F(s) = \int_s^1 \frac{1}{t} \left( 1 - \frac{1}{t} \right)^{n-1} d\tau$. Then the associated functional $F(\lambda, \cdot)$ admits a radial critical point $u_0 > 0$ if and only if $\lambda = (n\alpha_n)^{-1}$. Furthermore, for each $\lambda \leq (n\alpha_n)^{-1}$ we have

$$F(\lambda, u) \geq F\left( [n\alpha_n]^{-1}, u_0 \right) = 0, \quad \forall u \in W^{1,n}_0(\Omega), u \geq 0. \quad (1.12)$$

Another generalization is obtained by considering functions $f$ that satisfy the condition:

$$s \mapsto (n\alpha_n)^{-1} \left( 1 - \frac{1}{s} \right)^{n-1} \frac{1}{sf(s)} \quad \text{is strictly increasing in } (1, \infty). \quad (1.13)$$

For such nonlinearities and if $\Omega$ is a ball, Problem (1.10) admits radial positive solutions if and only if $\lambda$ belongs to the range of the map defined in (1.13). Under the additional requirement (1.9) we can prove that this range is the interval $(0, [n\alpha_n]^{-1})$ and explicitly calculate the infimum of the functional (1.8) when $\lambda \uparrow (n\alpha_n)^{-1}$. As a consequence both inequalities (1.4) and (1.7) can be generalized as follows:
**Theorem 1.3.** Assume that (1.9), (1.13) hold and let \( B \) be a ball of \( \mathbb{R}^n \). Then for any \( u \in W^{1,n}_0(\Omega) \) it holds

\[
\mathcal{F}(\lceil n\alpha_n \rceil^{-1}, u) > \inf_{u \in W^{1,n}_0(B)} \mathcal{F}(\lceil n\alpha_n \rceil^{-1}, u) = -n\alpha_n^{-1} \sum_{k=1}^{n-1} \frac{1}{k}.
\]

(1.14)

In the ball \( B \) the functional \( \mathcal{F}(\lambda, \cdot) \) admits for each \( \lambda < (n\alpha_n)^{-1} \) a unique minimizer, whereas the infimum in (1.14) is not achieved.

Our paper is organized as follows. Section 2 provides an alternative proof of the Pohozaev identity for radial functions solving \( -\Delta v = g(v) \). By applying this identity to the particular nonlinearity \( g(s) = e^s \), we are able in Section 3 to classify the radial solutions of the quasilinear equation \( -\Delta v = \pm e^v \) in \( \mathbb{R}^n \). In Section 4 we use a suitable substitution to reduce the nonlocal equation (1.10) to the local equation \( -\Delta v = \pm e^v \) and to classify the solutions of the nonlocal equation (1.10). In Section 5, after relating \( \mathcal{F}(\lambda, u) \) to the Moser-Trudinger inequality, we prove the lower bounds stated in Theorems 1.2 and 1.3.

## 2 Pohozaev identity for radial solutions

Given a ball \( B = B(0, R) \) of \( \mathbb{R}^n \) and a function \( g \in C^0(\mathbb{R}) \), we assume in this section the existence of a solution \( v \) to the problem

\[
-\Delta v = g(v), \quad v \in C^1(\bar{B}), \quad v \text{ radial},
\]

(2.1)

and derive some integral identities for \( v \).

For semilinear elliptic differential equations involving the Laplace operator, Pohozaev [20] derived an integral formula that has been widely used to prove non-existence of solutions. This identity has later been generalized by Pucci, Serrin [21] to \( C^2 \)-solutions of very general quasilinear elliptic equations. However, solutions satisfying non-linear equations involving the \( p \)-Laplace operator are not expected to be better than \( C^{1,\alpha} \) (see [26]). The work of Degiovanni et al. [8] has relaxed the \( C^2 \)-assumption and extended the Pucci-Serrin identity by assuming only the solution to be of class \( C^1(\bar{\Omega}) \). For Problem (2.1), by setting \( G(s) := \int_0^s g(t)dt \), this general identity reads

\[
\int_{B_r} G(v)dx - G(v(r)) = \frac{n-1}{n} \left| \frac{dv}{dr} \right|^n(r).
\]

(2.2)
Henceforth we set
\[ \omega = \text{measure of the unit sphere } S^{n-1} \subset \mathbb{R}^n, \]
and give an independent proof of (2.2) in the following equivalent form. Note that by integrating (2.1) one can easily check that (2.2) and (2.3) are equivalent.

**Proposition 2.1.** Let \( g \in C^0(\mathbb{R}) \) and set \( G(s) := \int_0^s g(t)dt \). Then any solution \( v \) of Problem (2.1) satisfies the following identities:

\[ \frac{\omega}{n} r^n G(v(r)) = \int_{B_r} G(v) \, dx - \frac{n-1}{n^2} \omega^{\frac{1}{n-1}} \left| \int_{B_r} g(v) \, dx \right|^{\frac{n}{n-1}}, \quad (2.3) \]

\[ \frac{r}{n} \frac{d}{dr} \left\{ \int_{B_r} G(v) \, dx \right\} = \int_{B_r} G(v) \, dx - \frac{n-1}{n^2} \omega^{\frac{1}{n-1}} \left| \int_{B_r} g(v) \, dx \right|^{\frac{n}{n-1}}. \quad (2.4) \]

**Proof:** On each ball \( B_r := B(0,r) \subset B \) define
\[ G(r) := \int_{B_r} g(v) \, dx, \quad r \in [0,R]. \]
Since \( g \circ v \in L^\infty(B) \) we easily check that the function \( G : [0,R] \rightarrow \mathbb{R} \) is Lipschitz continuous on \([0,R] \). Therefore \( r \mapsto |G(r)| \) is Lipschitz continuous, absolutely continuous and differentiable a.e in \([0,R] \).

Integrating (2.1) on the ball \( B_r \) and applying the divergence Theorem yield
\[ \int_{\partial B_r} (-|\nabla v|^{n-2} \nabla v, \nu) \, d\sigma = \int_{B_r} g(v) \, dx, \]
where \( \nu \) denotes the outward normal derivative on \( \partial B_r \). Since \( v \) is radial we deduce
\[ G(r) = \left| \frac{dv}{dr} \right|^{n-2} \left( -\frac{dv}{dr} \right) |\partial B_r| \quad \text{and} \quad \text{sgn}(G) = \text{sgn} \left( -\frac{dv}{dr} \right), \quad (2.5) \]
where by definition \( \text{sgn}(h)(r) := \frac{h(r)}{|h(r)|} \) if \( h(r) \neq 0 \), and \( \text{sgn}(h)(r) \) is zero otherwise. Moreover
\[ \frac{d|G|}{dr}(r) = \frac{d}{dr} \left| \int_{B_r} g(v) \, dx \right| = \text{sgn}(G) \int_{\partial B_r} g(v) \, d\sigma \]
\[ = \text{sgn} \left( -\frac{dv}{dr} \right) g(v(r)) |\partial B_r|. \quad (2.6) \]
Hence by (2.5) and (2.6) we obtain
\[
\frac{n-1}{n} \frac{d|\mathcal{G}|^{\frac{1}{n-1}}}{dr}(r) = |\mathcal{G}|^{\frac{1}{n-1}} \frac{d|\mathcal{G}|}{dr}(r)
\]
\[
= \left| \frac{dv}{dr} \right| |\partial B_r|^{\frac{1}{n-1}} \text{sgn} \left( -\frac{dv}{dr} \right) g(v) |\partial B_r|
\]
\[
= -\frac{dv}{dr} g(v) |\partial B_r|^{\frac{n}{n-1}}.
\]
Therefore,
\[
\frac{n-1}{n} \frac{d|\mathcal{G}|^{\frac{1}{n-1}}}{dr}(r) = -\omega^{\frac{n}{n-1}} r^{\frac{n}{n-1}} \frac{d}{dr} |G(v)|(r). \quad (2.7)
\]
By integrating equation (2.7) on the interval \([0, r]\) and recalling that \(|\mathcal{G}|\) is absolutely continuous we get
\[
\frac{n-1}{n} \int_{B_r} g(v) \, dx \bigg|^{\frac{n}{n-1}} = \omega^{\frac{n}{n-1}} \left\{ -G(v)r^n + n \int_0^r G(v)\rho^{n-1} \, d\rho \right\}
\]
\[
= \omega^{\frac{n}{n-1}} \left\{ -G(v)r^n + \frac{n}{\omega} \int_{B_r} G(v) \, dx \right\}. \quad (2.8)
\]
From (2.8) we immediately obtain (2.3). Equality (2.4) is now a consequence of (2.3) by noting that:
\[
G(v(r)) = \frac{1}{\omega r^{n-1}} \int_{\partial B_r} G(v) \, d\sigma = \frac{1}{\omega r^{n-1}} \frac{d}{dr} \int_{B_r} G(v) \, dx.
\]

\[\square\]

**Remark 2.2.** In dimension two and with the aim of deriving a priori estimates for radial subsolutions of \(-\Delta v \leq e^v\), Bandle \cite{Bandle} or Suzuki \cite{Suzuki} derive first a differential inequality for the function \(\int_{B_r} e^v \, dx\). In the proof of Proposition 2.1 we have extended this basic idea to higher dimension and general nonlinearity that may change sign.

From now on, we find it more convenient to rewrite (2.3) and (2.4) in terms of the constant \(\alpha_n\) defined in (1.3):
\[
\frac{\omega}{n} r^n G(v(r)) = \int_{B_r} G(v) \, dx - (n\alpha_n)^{\frac{1}{n-1}} \left| \int_{B_r} g(v) \, dx \right|^{\frac{n}{n-1}}, \quad (2.9)
\]
\[
\frac{r}{n} \frac{d}{dr} \left\{ \int_{B_r} G(v) \right\} = \int_{B_r} G(v) \, dx - (n\alpha_n)^{\frac{1}{n-1}} \left| \int_{B_r} g(v) \, dx \right|^{\frac{n}{n-1}}. \quad (2.10)
\]
Notice that when the function \(G\) is a multiple of \(g\), i.e. when \(g(s) = \lambda e^s\), equation (2.10) gives an ODE for the function \(\mathcal{G}(r) = \int_{B_r} g(v) \, dx\). This property will be exploited in the next section.
3 Radial solutions for a quasilinear Liouville equation

In this section we consider the special case of (1.10) in which \( \lambda \Omega f \equiv \pm 1 \) in a ball \( B \) centered at the origin and restrict the study to the class of radial solutions, i.e. we now study

\[
\begin{align*}
-\Delta_n v &= \varepsilon e^v, \quad \varepsilon = \pm 1, \\
v &\in W^{1,n}(B), \quad v \text{ radial.}
\end{align*}
\]

(3.1)

In the next section we shall reduce the general non-local Problem (1.10) to the local equation (3.1).

In dimension two the nonlinear equation in (3.1) is also called “Liouville equation”, because in [16] Liouville gave a representation formula of the solutions in terms of meromorphic functions (on any simply connected domain). When \( n \geq 2 \) the study of (3.1) (with \( \varepsilon = 1 \)) can be found in Clément et al. [[7], Section 6], where the solutions of (3.1) with zero Dirichlet data are explicitly given. Using a different approach, we shall see how the classification of solutions to (3.1) can be obtained directly from the Pohozaev identity (2.10). We start with the following result:

**Proposition 3.1.** Let \( v \) be a solution of (3.1) and set \( M(r) := \int_{B_r} e^v dx \) for \( r \leq R \). Then the following relations hold

\[
\begin{align*}
e^v(r) &= \frac{M(r)}{|B_r|} \left( 1 - \varepsilon \left[ n \alpha_n M(r) \right]^{\frac{1}{n-1}} \right), \quad (3.2) \\
e^v(0) &= \frac{M(r)}{|B_r|} \left( \frac{1}{1 - \varepsilon [n \alpha_n M(r)]^{\frac{1}{n-1}}} \right)^{n-1}, \quad \forall r \in [0, R], \quad (3.3) \\
M(r) &= \left| B_r \right| \frac{e^{v(0)}}{(1 + \varepsilon [n \alpha_n e^{v(0)} |B_r|^\frac{1}{n-1}]^{\frac{1}{n-1}})^{n-1}}, \quad \forall r \in [0, R]. \quad (3.4)
\end{align*}
\]

As a consequence

\[
e^{v(0)} e^{(n-1)\varepsilon(R)} = \left( \int_B e^v dx \right)^n. \quad (3.5)
\]

**Proof:** Equation (3.2) follows from applying (2.9) to \( g(s) = G(s) = \varepsilon e^s \). In order to prove (3.3) we first use (2.10) to obtain

\[
(n \alpha_n)^{\frac{1}{n-1}} \frac{r}{n} \frac{dM}{dr} = M \left\{ (n \alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} \right\},
\]

8
or equivalently
\[
\frac{(n\alpha_n)^{\frac{1}{n-1}}}{M \left\{ (n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} \right\}} \frac{dM}{dr} = \frac{n}{r}.
\] (3.6)

By noting that
\[
\frac{(n\alpha_n)^{\frac{1}{n-1}}}{M \left\{ (n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} \right\}} = \frac{1}{M} + \frac{\varepsilon M^{\frac{1}{n-1}-1}}{(n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}}},
\]
the differential equation (3.6) can explicitly be integrated on the interval \((r_0, r)\).

We then derive:
\[
\log \left( \frac{M(r)}{M(r_0)} \right) - (n - 1) \log \left( \frac{(n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} (r)}{(n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} (r_0)} \right) = \log \left( \frac{r}{r_0} \right)^n,
\]
and therefore
\[
M(r) \left( \frac{(n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} (r)}{(n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} (r_0)} \right)^{n-1} = \left( \frac{r}{r_0} \right)^n M(r_0).
\] (3.7)

By sending \(r_0\) to 0 in (3.7) we get
\[
M(r) \left( \frac{\omega}{n} r e^{\nu(0)} \right) = \frac{\omega}{n} r e^{\nu(0)}, \quad \forall r \in [0, R],
\]
which implies
\[
\frac{M^{\frac{1}{n-1}} (r)}{(n\alpha_n)^{\frac{1}{n-1}} - \varepsilon M^{\frac{1}{n-1}} (r)} = (\alpha_n \omega)^{\frac{1}{n-1}} e^{\nu(0)} r^{\frac{1}{n-1}}.
\] (3.8)

Equation (3.8) readily gives (3.3) and implies furthermore
\[
M^{\frac{1}{n-1}} (r) = (n\alpha_n)^{\frac{1}{n-1}} \frac{(\alpha_n \omega e^{\nu(0)})^{\frac{1}{n-1}} r^{\frac{n}{n-1}}}{1 + \varepsilon (\alpha_n \omega e^{\nu(0)})^{\frac{1}{n-1}} r^{\frac{n}{n-1}}}.
\]

The conclusion (3.4) follows from this last equality. Finally relation (3.5) is a direct consequence of (3.3) applied at \(r = R\) together with (3.2). \(\square\)
Based on our previous results, we have the following classification result:

**Proposition 3.2.** The radial solutions to (3.1) are given by the 1-parameter family:

$$v(x) = \log \frac{(\alpha_n \omega)^{-1} \mu^{n-1}}{(1 + \varepsilon \mu |x|^\frac{n}{n-1})^n}, \quad \mu > 0. \quad (3.9)$$

If $\varepsilon = -1$, the solution exists in $\bar{B}$ if and only if

$$n \alpha_n |B| e^{v(0)} < 1. \quad (3.10)$$

Furthermore the $n$-Dirichlet integral of $v$ is given by

$$\int_B |\nabla v|^n = \frac{1}{\alpha_n} \left| \int_1^{1 + \varepsilon \mu \frac{R}{n-1}} \frac{1}{\tau} \left( 1 - \frac{1}{\tau} \right)^{n-1} d\tau \right|. \quad (3.11)$$

**Proof:** Let $v$ be a solution of (3.1). Then by plugging (3.4) in (3.2) we see that $v$ has the form (3.9). Moreover, a straightforward calculation shows that the functions defined by (3.9) solve (3.1), which in radial coordinates reads:

$$-(n-1) \left| \frac{dv}{dr} \right|^{n-2} \left( \frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \right) = \varepsilon e^v.$$

We can now explicitly calculate the Dirichlet integral $\int_B |\nabla v|^n dx$ of any radial solution of (3.1). Using (3.9) we get

$$\frac{dv}{dr} = -\frac{n^2}{n-1} \frac{\varepsilon \mu r^{\frac{n}{n-1}}}{1 + \varepsilon \mu r^{\frac{n}{n-1}}}.$$

We then have

$$\int_B |\nabla v|^n dx = \omega \left( \frac{n^2}{n-1} \right)^n \int_0^R \frac{\varepsilon \mu r^{\frac{n}{n-1}}}{1 + \varepsilon \mu r^{\frac{n}{n-1}}} \frac{dr}{r}. \quad (3.12)$$

To calculate the integral in (3.12), we make the change of variable $t = 1 + \varepsilon \mu r^{\frac{n}{n-1}}$. Then a straight calculation yields

$$\int_0^R \left| \frac{\varepsilon \mu r^{\frac{n}{n-1}}}{1 + \varepsilon \mu r^{\frac{n}{n-1}}} \right|^n \frac{dr}{r} = \frac{n-1}{n} \int_1^{1 + \varepsilon \mu \frac{R}{n-1}} \left| \frac{\tau - 1}{\tau} \right|^n d\tau \frac{1}{\tau - 1} = \frac{n-1}{n} \left| \int_1^{1 + \varepsilon \mu \frac{R}{n-1}} \frac{1}{\tau} \left( 1 - \frac{1}{\tau} \right)^{n-1} d\tau \right|. \quad (3.13)$$

Using (3.12), (3.13) and the definition of $\alpha_n$ in (1.3) we obtain (3.11).
4 Application to a non-local equation on a ball

Given $f : (0, \infty) \to \mathbb{R}$ and a ball $B = B(0, R)$ in $\mathbb{R}^n$, the results obtained so far will be applied to the non-local problem:

$$-\Delta_n u = \frac{\lambda}{|B|} f \left( \int_B e^u \, dx \right) e^u, \quad u \in W^{1,n}_0(B), \ u \text{ radial.} \quad (4.1)$$

When the function $f$ satisfies suitable growth assumptions, the above problem is the Euler-Lagrange equation of the functional $(1.8)$. The present section will be helpful later to study the infimum of this functional.

Note first that the Moser-Trudinger inequality (1.2) and the regularity result of [26] imply that the solutions satisfying (4.1) are $C^1(\bar{B})$. Furthermore, by the maximum principle the solutions of (4.1) cannot change sign, and by applying the strong maximum principle we see that for a solution $u$ of (4.1) the following alternative holds:

$$\begin{cases} 
(a) & \lambda f(\int_B e^u \, dx) > 0 \quad \text{and} \quad u > 0, \\
(b) & \lambda f(\int_B e^u \, dx) < 0 \quad \text{and} \quad u < 0, \\
(c) & \lambda f(\int_B e^u \, dx) = 0 \quad \text{and} \quad u \equiv 0.
\end{cases} \quad (4.2)$$

Note that in the cases (a) and (b) the function

$$v := u + \log \left( \frac{1}{|B|} \lambda f(\int_B e^u \, dx) \right) \quad (4.3)$$

solves Problem (3.1) with $\varepsilon = \text{sgn}(u)$. We have actually the following result which is a more precise statement of Theorem 1.1 stated in the introduction.

**Theorem 4.1.** Let $u$ be a solution of Problem (4.1). Then $\sigma := \int_B e^u \, dx$ satisfies

$$n\alpha_n \lambda f(\sigma) = \frac{\varepsilon}{\sigma} \left( 1 - \frac{1}{\sigma} \right)^{n-1} \quad \text{with} \quad \varepsilon := \text{sgn}(\sigma - 1). \quad (4.4)$$

Conversely,

(a) if there exists $\sigma_0 \in (1, \infty)$ satisfying (4.4) then Problem (4.1) admits a positive solution,

(b) if there exists $\sigma_0 \in (0, 1)$ satisfying (4.4) then Problem (4.1) admits a negative solution.
(c) if \( \sigma_0 = 1 \) solves (4.4) then \( u \equiv 0 \) is a solution of (4.1).

In cases (a), (b) the solution is \( u = v - v(R) \) where \( v \) is given by (3.9) with \( \mu = |\sigma_0 - 1| \).

**Proof:** Assume Problem (4.1) admits a solution \( u \). If \( u \equiv 0 \) solves (4.1), then \( \sigma = 1 \), \( f(1) = 0 \) and identity (4.4) holds in this case. Assume now \( u \not\equiv 0 \). Applying (2.3) with \( G(s) = g(s) = \frac{\lambda}{|B|} f(\sigma) e^s \) we get

\[
\lambda f(\sigma) = \lambda f(\sigma) \sigma - (n \alpha_n)^{\frac{1}{n-1}} |\lambda f(\sigma)\sigma|^{\frac{n}{n-1}}. \tag{4.5}
\]

Since \( u \not\equiv 0 \) equation (4.1) shows that \( \lambda f(\sigma) \not= 0 \), and therefore (4.5) implies:

\[
n \alpha_n \frac{|\lambda f(\sigma)|^n}{(\lambda f(\sigma))^{n-1}} = \frac{1}{\sigma} \left( 1 - \frac{1}{\sigma} \right)^{n-1},
\]

which is equivalent to

\[
n \alpha_n |\lambda f(\sigma)| = \left( \frac{|\lambda f(\sigma)|}{\lambda f(\sigma)} \right)^{n-1} \frac{1}{\sigma} \left( 1 - \frac{1}{\sigma} \right)^{n-1}. \tag{4.6}
\]

Using (4.2) note that \( \text{sgn}(\lambda f(\sigma)) = \text{sgn}(u) = \text{sgn}(\sigma - 1) \). Therefore (4.6) yields:

\[
n \alpha_n \varepsilon \lambda f(\sigma) = \frac{1}{\sigma} \left| 1 - \frac{1}{\sigma} \right|^{n-1}.
\]

Hence (4.4) also holds for \( u \not\equiv 0 \).

Conversely, assume that (4.4) admits a solution \( \sigma_0 \in (0, \infty) \setminus \{1\} \). By setting \( \varepsilon := \text{sgn}(\sigma_0 - 1) \) consider a solution \( v \) of the problem:

\[
-\Delta_n v = \varepsilon e^v, \quad e^{v(0)} = \frac{|\sigma_0 - 1|^{n-1}}{n \alpha_n |B|}. \tag{4.7}
\]

By Proposition 3.2 such a function \( v \) exists and \( v \in C^1(\bar{B}) \) (since condition (3.10) holds). We claim that

\[
u(r) := v(r) - v(R) \tag{4.8}
\]

solves Problem (4.1). Indeed we have

\[
-\Delta_n u = \varepsilon e^{v(R)} e^u, \quad u \in W_0^{1,n}(B),
\]
and so we only need to verify that
\[ \varepsilon e^v(R) = \frac{\lambda}{|B|} \int_B e^{v(v(R))} \, dx. \] (4.9)

Using (3.2) and (3.4), we write \( e^v(R) \) and \( \int_B e^v \, dx \) as a function of \( e^v(0) \), which can in turn be expressed in terms of \( \sigma_0 \) using (4.7):

\[ \varepsilon e^v(R) = \varepsilon e^{v(0)} \left( 1 + \varepsilon \left[ n\alpha_n |B| e^{v(0)} \right]^\frac{1}{n-1} \right)^n = \varepsilon \frac{1}{n\alpha_n |B| \sigma_0} \left( 1 - \frac{1}{\sigma_0} \right)^{n-1}, \]
\[ e^{-v(R)} \int_B e^v = 1 + \varepsilon \left[ n\alpha_n |B| e^{v(0)} \right]^\frac{1}{n-1} = 1 + \varepsilon |\sigma_0 - 1| = \sigma_0. \]

Therefore (4.9) is equivalent to
\[ \varepsilon \frac{1}{\sigma_0} \left( 1 - \frac{1}{\sigma_0} \right)^{n-1} = n\alpha_n \lambda f(\sigma_0). \]

Finally if \( \sigma_0 = 1 \) is a solution of (4.4) then either \( \lambda = 0 \) or \( f(1) = 0 \), and in both cases \( u \equiv 0 \) solves (4.1). \( \Box \)

Theorem 4.1 shows that radial solutions of the Dirichlet Problem (4.1) are completely classified by the real numbers \( \sigma_0 > 0 \) solving (4.4). For \( \sigma_0 \in (0, \infty) \setminus \{1\} \) using (4.7), (4.8) and (3.9), the solutions of (4.1) are explicitly given by:
\[ u(x) = n \log \left( \frac{\sigma_0}{1 + [\sigma_0 - 1] |x/R|^{\frac{n}{n-1}}} \right), \quad |x| < R. \] (4.10)

For example in the case
\[ f(s) = \frac{1}{s} \left[ 1 - \frac{1}{s} \right]^{k-1} \left( 1 - \frac{1}{s} \right), \quad k \geq 1, \] (4.11)
the function \( u \equiv 0 \) trivially solves (4.1) and we can prove:

**Proposition 4.2.** Consider Problem (4.1) with \( f \) given by (4.11).

(a) For \( k = n - 1 \), Problem (4.1) admits a non-trivial solution if and only if \( \lambda = (n\alpha_n)^{-1} \). In this case the family of positive (resp. negative) solutions is given by (4.10) with \( \sigma_0 \in (1, \infty) \) (resp. \( \sigma_0 \in (0, 1) \)).
(b) For \( k \neq n - 1 \), Problem (4.1) admits positive solutions if and only if
\[
\begin{cases}
1 \leq k < n - 1 \quad \text{and} \quad \lambda \in (0, (n \alpha_n)^{-1}), \\
\text{or} \quad k > n - 1 \quad \text{and} \quad \lambda \in ((n \alpha_n)^{-1}, \infty),
\end{cases}
\]
and in this case the positive solution is unique.

(c) For \( k \neq n - 1 \), Problems (4.1) admits negative solutions if and only if \( \lambda > 0 \), and in this case the negative solution is unique.

**Proof:** Assume the existence of a solution \( u \neq 0 \). For the function \( f \) we are considering, equation (4.4) reads:
\[
n \alpha_n \lambda = \left| 1 - \frac{1}{\sigma} \right|^{n-1-k}, \quad \sigma \in (0, \infty) \setminus \{1\}.
\]
(4.12)

(a) If \( k = n - 1 \) equation (4.12) is solvable if and only if \( \lambda = (n \alpha_n)^{-1} \). In this case any \( \sigma \in (0, \infty) \setminus \{1\} \) is a solution, which is positive (resp. negative) if \( \sigma > 1 \) (resp. \( \sigma < 1 \)).

(b) If \( 1 \leq k < n - 1 \), (4.12) admits a solution \( \sigma > 1 \) (which is unique) if and only if \( n \alpha_n \lambda \in (0, 1) \). Whereas for \( k > n - 1 \), (4.12) admits a solution \( \sigma > 1 \) (again unique) if and only if \( n \alpha_n \lambda > 1 \).

(c) Finally we easily see that (4.12) admits a solution \( \sigma \in (0, 1) \) for each \( \lambda > 0 \) which is unique whenever \( k \neq n - 1 \).

We conclude by applying Theorem 4.1. \( \square \)

Another immediate consequence of Theorem 4.1 is the following:

**Proposition 4.3.** Assume that the mapping \( f_0 : (1, \infty) \to \mathbb{R} \) defined by
\[
s \mapsto f_0(s) := \frac{(n \alpha_n)^{-1}}{s f(s)} \left( 1 - \frac{1}{s} \right)^{n-1},
\]
(4.13)
is strictly monotone. Then Problem (4.1) admits a positive solution if and only if \( \lambda \in \text{Range}(f_0) \) and in this case the solution is unique.

If condition (4.13) is stated on the interval \((0, 1)\) then we get a similar result for the existence of negative solutions. An example of a function \( f \) satisfying (4.13) is given by \( f(s) = \frac{1}{s} \), for which Problem (4.1) reads
\[
-\Delta_n u = \lambda \frac{e^u}{\int_B e^u \, dx}, \quad u \in W^{1,n}_0(B).
\]
(4.14)
For this nonlinearity $f$ and for each $\lambda$ equation (4.4) admits the unique solution:

$$\sigma_\lambda = \left(1 - \varepsilon [n\alpha_n|\lambda|]^{\frac{1}{n-1}}\right)^{-1}.$$  

Thus Theorem 4.1 shows that Problem (4.14) admits a positive solution (resp. negative) iff $\lambda \in (0, (n\alpha_n)^{-1})$ (resp. $\lambda < 0$). The solution is unique and referring to (4.10), it can explicitly be written in term of the parameter $\lambda$:

$$u_\lambda(x) = -n \log \left(1 - \varepsilon [n\alpha_n|\lambda|]^{\frac{1}{n-1}} + \varepsilon [n\alpha_n|\lambda|]^{\frac{1}{n-1}} \left|\frac{x}{R}\right|^{\frac{n}{n-1}}\right). \tag{4.15}$$

When $n = 2$ the solution (4.15) becomes

$$u_\lambda(x) = \log \left(\frac{8\pi}{8\pi - \lambda + \lambda|x/R|^2}\right)^2, \tag{4.16}$$

which can already be found in Aly [1] or in Suzuki [[24], Theorem 3.1]. Let us emphasize that these authors obtained (4.16) by making a suitable change of variable, while our method of proof relies on the Pohozaev identity stated in Proposition 2.1 and applies to any dimension $n \geq 2$.

We conclude this section with

**Proposition 4.4.** Let $u$ be a solution of Problem (4.1) and set $\sigma := \int_B e^u \, dx$. Then

$$\int_B |\nabla u|^n \, dx = \frac{1}{\alpha_n} \left|\int_1^{\sigma} \frac{1}{\tau} \left(1 - \frac{1}{\tau}\right)^{n-1} \, d\tau\right|. \tag{4.17}$$

**Proof:** Consider $v$ defined by (4.3) which solves $-\Delta_n v = \text{sgn}(u)e^v$. By applying (3.3) we easily obtained

$$e^{u(0)} = \sigma^n. \tag{4.18}$$

Formula (4.17) is now obtained from (3.11) if we note that $\mu = (\alpha_n\omega e^{v(0)})^{\frac{n}{n-1}}$. Indeed, using successively (4.18) and (4.4), we get

$$\text{sgn}(u)\mu R^{\frac{n}{n-1}} = \text{sgn}(u)(\alpha_n\omega R^n e^{v(0)})^{\frac{1}{n-1}} = \text{sgn}(u) (n\alpha_n|\lambda f(\sigma)|e^{u(0)})^{\frac{1}{n-1}}$$

$$= \text{sgn}(u)(n\alpha_n|\lambda f(\sigma)|\sigma^n)^{\frac{1}{n-1}} = \text{sgn}(u) \left|\frac{1}{\sigma}\right| \sigma$$

$$= \text{sgn}(u)|\sigma - 1| = \sigma - 1. \tag{4.19}$$

Plugging (4.19) in (3.11) gives the identity (4.17).
5 Optimal constants

The aim of this section is to study the infimum of the functional $F(\lambda, \cdot)$ defined in (1.8). We start by recalling that the Moser-Trudinger inequality (1.1) implies for each $\lambda \in [0, (n\alpha_n)^{-1}]$ the inequality

$$\frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \lambda \log \left( \int_{\Omega} e^u \, dx \right) \geq - \frac{1}{n\alpha_n} \log C_n, \quad \forall u \in W_{0}^{1,n}(\Omega),$$

where the constant $C_n$ is the one appearing in (1.1). Indeed by using Young’s inequality $ab \leq \left( \frac{n}{n-1} \right)^{n-1} |a|^{n-1} |b|$, we get

$$|u| = \left( \frac{n-1}{n\mu_n} \right)^{\frac{n-1}{n}} \|\nabla u\|_n \left( \frac{n\mu_n}{n-1} \right)^{\frac{n-1}{n}} \|u\| \left( \frac{n-1}{n\mu_n} \right)^{\frac{n-1}{n}} \|\nabla u\|_n$$

$$\leq \frac{1}{n} \left( \frac{n-1}{n\mu_n} \right)^{n-1} \|\nabla u\|_n + \mu_n \left( \frac{|u|}{\|\nabla u\|_n} \right)^{\frac{n-1}{n}}.$$ 

Therefore, since $\mu_n = n\omega_1^{\frac{1}{n-1}}$ and using the constant $\alpha_n$ defined by (1.3), we deduce

$$\log \left( \int_{\Omega} e^u \, dx \right) \leq \alpha_n \|\nabla u\|_n + \log C_n.$$ 

Inequality (5.2) readily implies (5.1) as well as (1.2). We can slightly extend this result as follows:

Proposition 5.1. Let $\Omega \subset \subset \mathbb{R}^2$ and $F \in C^0(0, \infty)$ satisfying

$$\limsup_{s \to 0^+} F(s) < \infty \quad \text{and} \quad \limsup_{s \to \infty} \{F(s) - \log s\} < \infty. \quad (5.3)$$

Then

(a) $F(\lambda, \cdot)$ is bounded from below for each $\lambda \in (0, (n\alpha_n)^{-1})$,

(b) $F(\lambda, \cdot)$ admits a minimizer for each $\lambda \in (0, (n\alpha_n)^{-1})$,

(c) $F(\lambda, \cdot)$ is unbounded from below for $\lambda > (n\alpha_n)^{-1}$.

Proof: Using (5.1) it is known that the mapping $W_{0}^{1,n}(\Omega) \to L^1(\Omega), \ u \mapsto e^u$ is compact. Claims (a) and (b) follow now easily. To prove the last statement we construct appropriate trial functions. Similarly to what has been done in [22] (on a two-dimensional torus), we consider for each $(a, \mu) \in \mathbb{R}^n \times [1, \infty)$ the functions

$$\delta_{a,\mu}(x) := \log \left( \frac{(\alpha_n \omega_1)^{-1} \mu^{n-1}}{1 + \mu |x - a|^{\frac{n}{n-1}}} \right).$$

(5.4)
As a consequence of Proposition 3.2 these functions are radial solutions of $-\Delta_n v = e^v$ on $\mathbb{R}^n$, and by letting $r \to \infty$ in (3.4) we see

$$\int_{\mathbb{R}^n} e^{\delta_{a,\mu}(x)} \, dx = (n\alpha_n)^{-1}. \quad (5.5)$$

Given a fixed ball $B(a, r) \subset \subset \Omega$, set $c_{a,\mu}$ to be the value of $\delta_{a,\mu}$ on $\partial B(a, r)$ and define the function

$$\tilde{\delta}_{a,\mu}(x) := \begin{cases} \delta_{a,\mu}(x) - c_{a,\mu} & \text{if } x \in B(a, r), \\ 0 & \text{otherwise.} \end{cases}$$

Using (5.5) and (3.11), we deduce

$$\log \left( \int_{\Omega} e^{\delta_{a,\mu}} \, dx \right) = \log \mu + O(1), \quad (5.6)$$

$$\int_{\Omega} |\nabla \tilde{\delta}_{a,\mu}|^n \, dx = n(n\alpha_n)^{-1} \log \mu + O(1). \quad (5.7)$$

In particular, we have

$$\frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \lambda \log \left( \int_{\Omega} e^u \, dx \right) = ((n\alpha_n)^{-1} - \lambda) \log \mu + O(1). \quad (5.8)$$

Hence as $\lambda > (n\alpha_n)^{-1}$ we see that (5.8) tends to $-\infty$ for $\mu \to \infty$. \qed

When the domain is a ball $B$, the energy $\mathcal{F}(\lambda, u)$ of any radially symmetric critical point $u$ can be expressed as a function of $\sigma := \int_{B} e^u \, dx$. Indeed by applying (4.17) and (4.4) we have

$$\mathcal{F}(\lambda, u) = (n\alpha_n)^{-1} \left| \int_{1}^{\sigma} \left(1 - \frac{1}{\tau}\right)^{n-1} \frac{d\tau}{\tau} \right| - \lambda F(\sigma)$$

$$= (n\alpha_n)^{-1} \left\{ \int_{1}^{\sigma} \left(1 - \frac{1}{\tau}\right)^{n-1} \frac{d\tau}{\tau} - \text{sgn}(\sigma - 1) \frac{1 - \frac{1}{\sigma}}{\sigma f(\sigma)} F(\sigma) \right\}. \quad (5.9)$$

We first consider the case when $F(s) = \int_{1}^{s} \frac{1}{\tau} \left(1 - \frac{1}{\tau}\right)^{n-1} \, d\tau$.

**Proof of Theorem 1.2:**

Proposition 4.2 applied to $f(s) = \frac{1}{s} \left(1 - \frac{1}{s}\right)^{n-1}$ shows that
(i) for $\lambda \neq (n\alpha_n)^{-1}$ the function $u \equiv 0$ is the unique non-negative radial critical point of $F(\lambda, \cdot)$,

(ii) $F([n\alpha_n]^{-1}, \cdot)$ admits a family of radial critical points $u_{\mu} > 0$ and (5.9) implies

$$F([n\alpha_n]^{-1}, u_{\mu}) = 0. \quad (5.10)$$

To prove (1.12) we define

$$\tilde{F}(s) = \begin{cases} 
\int_{1}^{s} \frac{1}{t} \left(1 - \frac{1}{t}\right)^{n-1} dt & \text{if } s \geq 1, \\
0 & \text{if } s \in (0, 1), 
\end{cases} \quad (5.11)$$

and denote by $\tilde{F}(\lambda, \cdot)$ the functional associated with (5.11). By Proposition 5.1, $\tilde{F}(\lambda, \cdot)$ admits a minimizer $u_{\lambda}$ for each $\lambda \in [0, (n\alpha_n)^{-1})$. We check easily that $u_{\lambda} \geq 0$ and therefore using Schwarz symmetrization we deduce that $u_{\lambda}$ must be radial. Then Proposition 4.2 shows that $u_{\lambda} \equiv 0$. Hence for each $\lambda < (n\alpha_n)^{-1}$ we get

$$\tilde{F}(\lambda, u_{\lambda}) \geq \tilde{F}(\lambda, 0) \geq 0.$$

By considering $\lambda \uparrow (n\alpha_n)^{-1}$ together with (5.10) we get (1.12). \qed

When $F$ satisfies the requirements (1.9) and (1.13) we can also calculate the infimum of the associated functional $F((n\alpha_n)^{-1}, \cdot)$:

**Proof of Theorem 1.3:**

Note first that our assumptions readily imply $f > 0$ and we claim:

$$L := \liminf_{s \to \infty} \left\{ (sf(s) - 1) \log s \right\} = 0, \quad (5.12)$$

$$\lim_{s \to \infty} sf(s) = \lim_{s \to \infty} \frac{(1 - \frac{1}{2})^{n-1}}{s f(s)} = 1. \quad (5.13)$$

To prove (5.12) assume $L > 0$. Then for $a > 1$ large enough we have:

$$F(s) - \log s = F(a) - \log a + \int_{a}^{s} \left( \frac{tf(t) - 1}{t \log t} \right) dt$$

$$> F(a) - \log a + \frac{L}{2} \int_{a}^{s} \frac{dt}{t \log t}, \quad \forall s > a.$$

The last inequality would then imply $\lim_{s \to \infty} \{F(s) - \log s\} = +\infty$, in contradiction to our assumption (1.9). If $L < 0$ we get a similar contradiction. Hence $L = 0$
and (5.12) is established. Concerning (5.13), we first note that both functions appearing in (5.13) admit and have the same limit as \( s \to \infty \). Indeed
\[
\lim_{s \to \infty} sf(s) = \lim_{s \to \infty} \frac{sf(s)}{(1 - \frac{1}{s})^{n-1}} \left(1 - \frac{1}{s}\right)^{n-1} = \lim_{s \to \infty} \frac{sf(s)}{(1 - \frac{1}{s})^{n-1}},
\]
and this latter limit exists due to (1.13) since the function \( sf(s) \left(1 - \frac{1}{s}\right)^{1-n} \) is decreasing and positive. Using (5.12) we immediately conclude \( \lim_{s \to \infty} sf(s) = 1 \), and (5.13) follows.

Let \( \lambda < (n\alpha_n)^{-1} \) and \( u_\lambda \) be a minimizer of the functional \( F(\lambda, \cdot) \) (which exists by Proposition 5.1). Using Schwarz symmetrization (see [12]), we may assume without loss of generality that \( \Omega \) is a ball \( B \) and \( u_\lambda \) is a radial positive function. Clearly for any \( u \in W^{1,n}_0(B) \) we have
\[
F((n\alpha_n)^{-1}, u) = \liminf_{\lambda \uparrow (n\alpha_n)^{-1}} F(\lambda, u) \geq \liminf_{\lambda \uparrow (n\alpha_n)^{-1}} F(\lambda, u_\lambda). \tag{5.14}
\]
By setting \( \sigma = \sigma(\lambda) := \int_B e^{u_\lambda} \, dx \), we remind the reader now that the value \( F(\lambda, u_\lambda) \) can be written in terms of \( \sigma \) as in (5.9). Hence
\[
n\alpha_n F(\lambda, u_\lambda) = \int_1^{\sigma} \frac{1}{\tau} \left(1 - \frac{1}{\tau}\right)^{n-1} d\tau - \frac{(1 - \frac{1}{\sigma})^{n-1}}{\sigma f(\sigma)} F(\sigma) \\
= \int_1^{\sigma} \left(1 - \frac{1}{\tau}\right)^{n-1} - 1 \right) d\tau - \left\{ \frac{(1 - \frac{1}{\sigma})^{n-1}}{\sigma f(\sigma)} F(\sigma) - \log \sigma \right\} \\
= \int_0^{1-\frac{1}{\sigma}} s^{n-1} - \frac{1}{1 - s} ds - \left\{ \frac{(1 - \frac{1}{\sigma})^{n-1}}{\sigma f(\sigma)} F(\sigma) - \log \sigma \right\}. \tag{5.15}
\]
Since the value \( \sigma \) satisfies (4.4), the monotonicity assumption (1.13) with the property (5.13) imply that \( \lim_{\lambda \uparrow (n\alpha_n)^{-1}} \sigma(\lambda) = \infty \). Therefore from (5.14) and (5.15) we get
\[
n\alpha_n F((n\alpha_n)^{-1}, u) \geq - \sum_{k=1}^{n-1} \frac{1}{k} + \liminf_{\lambda \uparrow (n\alpha_n)^{-1}} \left\{ \log \sigma - \frac{(1 - \frac{1}{\sigma})^{n-1}}{\sigma f(\sigma)} F(\sigma) \right\} \\
= - \sum_{k=1}^{n-1} \frac{1}{k}, \tag{5.16}
\]
where the last equality follows by using (5.12) and (1.9). With (5.16) the proof of Theorem 1.3 follows.\qed

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Remark 5.2. In the particular case when $F(s) = \log s$, Proposition 4.4 readily shows that any solution $(\lambda, u_\lambda)$ of (4.14) with $\lambda > 0$ satisfies

$$\sigma = \int_B e^{u_\lambda} \, dx = \left(1 - [n\alpha_n \lambda]^{\frac{1}{n-1}}\right)^{-1}. \quad (5.17)$$

In this case, by using (5.9), $F(\lambda, u_\lambda)$ can be expressed as a function of $\lambda$:

$$F(\lambda, u_\lambda) = \frac{1}{n} \int_B |\nabla u_\lambda|^n \, dx - \lambda \log \left(\int_B e^{u_\lambda} \, dx\right) - (n\alpha_n)^{-1}\left\{(1 - n\alpha_n \lambda) \log \left(1 - [n\alpha_n \lambda]^{\frac{1}{n-1}}\right) + \sum_{k=1}^{n-1} \frac{[n\alpha_n \lambda]^\frac{1}{n-1}}{k}\right\}.$$ 

In dimension $n = 2$ this result can be found in [1].

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