Complement to the text of Michelle Wachs

The excerpt from Wachs' text is fairly elementary, but it uses some notions from topology that are not prerequisites for this course. This complement aims to explain these notions in an informal way.

One of the major objectives of topology is the following: given topological spaces[1] X and Y, determine whether they are "equivalent", with some suitable definition of equivalence. Usually, the most restrictive notion one uses is that of spaces being *homeomorphic*:

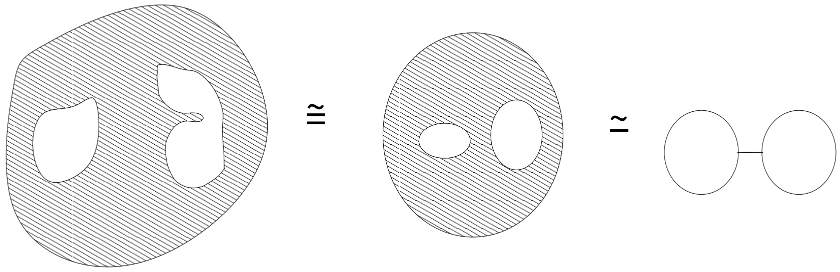
Definition. *A* homeomorphism *from X to Y is a bijection $f : X \to Y$ such that f and f^{-1} are both continuous. If such a map exists, X and Y are said to be* homeomorphic*, and we write $X \cong Y$.*

Intuitively, X and Y are homeomorphic if one can construct Y by stretching and bending X without ripping it apart.

A strictly weaker concept is that of *homotopy equivalence*. We do not define it formally here, but the intuition is that X and Y are homotopy equivalent, denoted $X \simeq Y$, if one can construct a space homeomorphic to Y from X by removing and/or gluing on solid pieces in a way which never introduces or fills in "holes" in the space.

A *contractible* space is one which is homotopy equivalent to a point. One simple class of contractible spaces is given by the *cones*; a simplicial complex Δ is a *cone* with *apex* v if v is a vertex which is included in every maximal face of Δ. (Intuitively, it should make sense that one can compress a cone onto its apex without changing its homotopy type.)

In the picture below, all three spaces are homotopy equivalent; the two leftmost ones are homeomorphic to each other but not to the third.



Generally, it is difficult to determine whether spaces are homeomorphic, or even homotopy equivalent. To attack the situation, people have invented various invariants that are preserved under these equivalences. One example is the (reduced) Euler characteristic. A finer invariant is given by the (reduced) *Betti numbers* of the space. Formally, the reduced Betti number $\widetilde{\beta}_i(\Delta)$ of a space is the rank of the i:th reduced homology group of Δ. Homology groups are beyond the scope of this course (hopefully, though, we found time to say something abut them during the lecture). However, for the simple spaces that we will consider, the following intuitive picture works. The i:th reduced Betti number is the number of $(i+1)$-dimensional "holes" in the space. As an example, the i-dimensional sphere has $\widetilde{\beta}_i = 1$ and $\widetilde{\beta}_j = 0$ if $j \neq i$. All three spaces in the picture above have $\widetilde{\beta}_1 = 2$, and all other $\widetilde{\beta}_i$ vanish.

[1]The only topological spaces that we consider here are compact subsets of Euclidean space $\mathbb{R}^n$