LECTURE 1
Basic definitions, results, and examples

1.1. Order complexes and face posets

We begin by defining the order complex of a poset and the face poset of a simplicial complex. These constructions enable us to view posets and simplicial complexes as essentially the same topological object. We shall assume throughout these lectures that all posets and simplicial complexes are finite, unless otherwise stated.

An *abstract simplicial complex* Δ on finite vertex set V is a nonempty collection of subsets of V such that

- $\{v\} \in \Delta$ for all $v \in V$
- if $G \in \Delta$ and $F \subseteq G$ then $F \in \Delta$.

The elements of Δ are called *faces* (or simplices) of Δ and the maximal faces are called *facets*. We say that a face F has dimension d and write $\dim F = d$ if $d = |F| - 1$. Faces of dimension d are referred to as d-faces. The *dimension* $\dim \Delta$ of Δ is defined to be $\max_{F \in \Delta} \dim F$. We also allow the (-1)-dimensional complex $\{\emptyset\}$, which we refer to as the *empty simplicial complex*. It will be convenient to refer to the empty set $\emptyset$, as the *degenerate empty complex* and say that it has dimension -2, even though we don't really consider it to be a simplicial complex. If all facets of Δ have the same dimension then Δ is said to be *pure*.

A d-dimensional *geometric simplex* in $\mathbb{R}^n$ is defined to be the convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$ called *vertices*. The convex hull of any subset of the vertices is called a *face* of the geometric simplex. A *geometric simplicial complex* K in $\mathbb{R}^n$ is a nonempty collection of geometric simplices in $\mathbb{R}^n$ such that

- Every face of a simplex in K is in K.
- The intersection of any two simplices of K is a face of both of them.

From a geometric simplicial complex K, one gets an abstract simplicial complex $\Delta(K)$ by letting the faces of $\Delta(K)$ be the vertex sets of the simplices of K. Every abstract simplicial complex Δ can be obtained in this way, i.e., there is a geometric simplicial complex K such that $\Delta(K) = \Delta$. Although K is not unique, the underlying topological space, obtained by taking the union of the simplices of K under the usual topology on $\mathbb{R}^n$, is unique up to homeomorphism. We refer to this space as the *geometric realization* of Δ and denote it by $\|\Delta\|$. We will usually

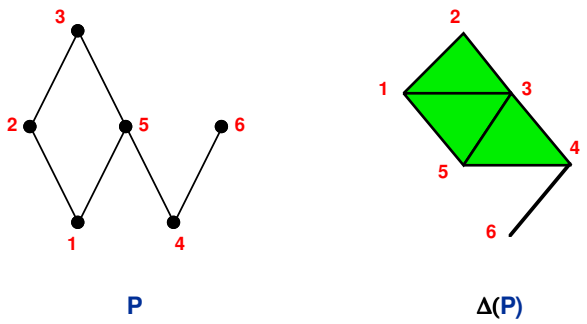

Figure 1.1.1. Order complex of a poset

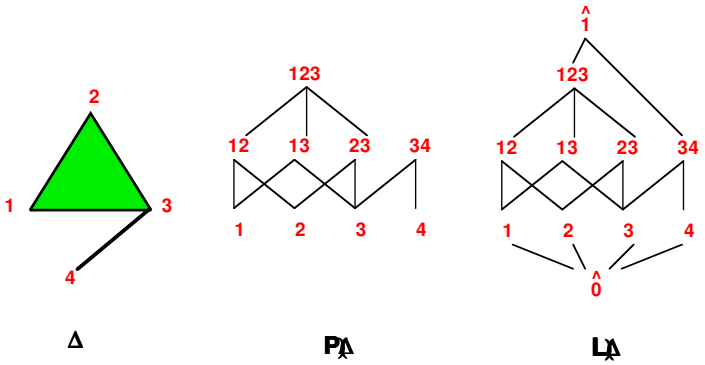

Figure 1.1.2. Face poset and face lattice of a simplicial complex

drop the $\| \ \|$ and let Δ denote an abstract simplicial complex as well as its geometric realization.

To every poset P, one can associate an abstract simplicial complex $\Delta(P)$ called the *order complex* of P. The vertices of $\Delta(P)$ are the elements of P and the faces of $\Delta(P)$ are the chains (i.e., totally ordered subsets) of P. (The order complex of the empty poset is the empty simplicial complex $\{\emptyset\}$.) For example, the Hasse diagram of a poset P and the geometric realization of its order complex are given in Figure 1.1.1.

To every simplicial complex Δ, one can associate a poset $P(\Delta)$ called the *face poset* of Δ, which is defined to be the poset of nonempty faces ordered by inclusion. The *face lattice* $L(\Delta)$ is $P(\Delta)$ with a smallest element $\hat{0}$ and a largest element $\hat{1}$ attached. An example is given in Figure 1.1.2.

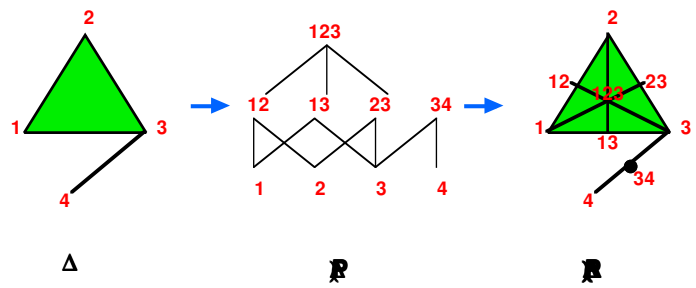

Figure 1.1.3. Barycentric subdivision

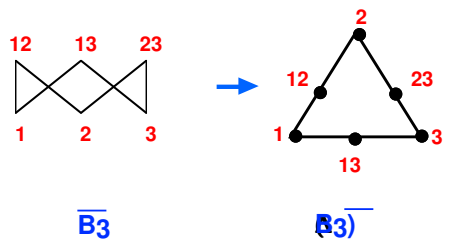

Figure 1.1.4. Order complex of the subset lattice (Boolean algebra)

If we start with a simplicial complex Δ, take its face poset $P(\Delta)$, and then take the order complex $\Delta(P(\Delta))$, we get a simplicial complex known as the *barycentric subdivision* of Δ; see Figure 1.1.3. The geometric realizations are always homeomorphic,

$$\Delta \cong \Delta(P(\Delta)).$$

When we attribute a topological property to a poset, we mean that the geometric realization of the order complex of the poset has that property. For instance, if we say that the poset P is homeomorphic to the n-sphere $\mathbb{S}^n$ we mean that $\|\Delta(P)\|$ is homeomorphic to $\mathbb{S}^n$.

Example 1.1.1. The Boolean algebra. Let B_n denote the lattice of subsets of $[n] := \{1, 2, \ldots, n\}$ ordered by containment, and let $\bar{B}_n := B_n - \{\emptyset, [n]\}$. Then

$$\bar{B}_n \cong \mathbb{S}^{n-2}$$

because $\Delta(\bar{B}_n)$ is the barycentric subdivision of the boundary of the $(n-1)$-simplex. See Figure 1.1.4.

We now review some basic poset terminology. An m-chain of a poset P is a totally ordered subset $c = \{x_1 < x_2 < \cdots < x_{m+1}\}$ of P. We say the length $l(c)$ of c is m. We consider the empty chain to be a (-1)-chain. The length $l(P)$ of P is

defined to be

$$l(P) := \max\{l(c) : c \text{ is a chain of } P\}.$$

Thus, $l(P) = \dim \Delta(P)$ and $l(P(\Delta)) = \dim \Delta$.

A chain of P is said to be *maximal* if it is inclusionwise maximal. Thus, the set $\mathcal{M}(P)$ of maximal chains of P is the set of facets of $\Delta(P)$. A poset P is said to be *pure* (also known as ranked or graded) if all maximal chains have the same length. Thus, P is pure if and only if $\Delta(P)$ is pure. Also a simplicial complex Δ is pure if and only if its face poset $P(\Delta)$ is pure. The posets and simplicial complexes of Figures 1.1.1 and 1.1.2 are all nonpure, while the poset and simplicial complex of Figure 1.1.4 are both pure.

For $x \leq y$ in P, let (x, y) denote the open interval $\{z \in P : x < z < y\}$ and let $[x, y]$ denote the closed interval $\{z \in P : x \leq z \leq y\}$. Half open intervals $(x, y]$ and $[x, y)$ are defined similarly.

If P has a unique minimum element, it is usual to denote it by $\hat{0}$ and refer to it as the *bottom* element. Similarly, the unique maximum element, if it exists, is denoted $\hat{1}$ and is referred to as the *top* element. Note that if P has a bottom element $\hat{0}$ or top element $\hat{1}$ then $\Delta(P)$ is contractible since it is a cone. We usually remove the top and bottom elements and study the more interesting topology of the remaining poset. Define the *proper part* of a poset P, for which $|P| > 1$, to be

$$\bar{P} := P - \{\hat{0}, \hat{1}\}.$$

In the case that $|P| = 1$, it will be convenient to define $\Delta(\bar{P})$ to be the degenerate empty complex $\emptyset$. We will also say $\Delta((x, y)) = \emptyset$ and $l((x, y)) = -2$ if $x = y$.

For posets with a bottom element $\hat{0}$, the elements that cover $\hat{0}$ are called *atoms*. For posets with a top element $\hat{1}$, the elements that are covered by $\hat{1}$ are called *coatoms*.

A poset P is said to be *bounded* if it has a top element $\hat{1}$ and a bottom element $\hat{0}$. Given a poset P, we define the bounded extension

$$\hat{P} := P \cup \{\hat{0}, \hat{1}\},$$

where new elements $\hat{0}$ and $\hat{1}$ are adjoined (even if P already has a bottom or top element).

A poset P is said to be a *meet semilattice* if every pair of elements $x, y \in P$ has a meet $x \wedge y$, i.e. an element less than or equal to both x and y that is greater than all other such elements. A poset P is said to be a *join semilattice* if every pair of elements $x, y \in P$ has a join $x \vee y$, i.e. a unique element greater than or equal to both x and y that is less than all other such elements. If P is both a join semilattice and a meet semilattice then P is said to be a *lattice*. It is a basic fact of lattice theory that any finite meet (join) semilattice with a top (bottom) element is a lattice.

The *dual* of a poset P is the poset P^* on the same underlying set with the order relation reversed. Topologically there is no difference between a poset and its dual since $\Delta(P)$ and $\Delta(P^*)$ are identical simplicial complexes. The direct product $P \times Q$ of two posets P and Q is the poset whose underlying set is the cartesian product $\{(p, q) : p \in P, q \in Q\}$ and whose order relation is given by

$$(p_1, q_1) \leq_{P \times Q} (p_2, q_2) \text{ if } p_1 \leq_P p_2 \text{ and } q_1 \leq_Q q_2.$$

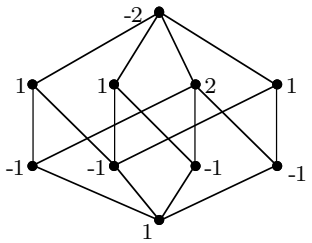

Figure 1.2.1. $\mu(\hat{0}, x)$

Define the *join* of two simplicial complexes Δ and Γ on disjoint vertex sets to be the simplicial complex given by

$$\Delta * \Gamma := \{A \cup B : A \in \Delta, B \in \Gamma\}. \tag{1.1.1}$$

The *join* (or ordinal sum) $P * Q$ of posets P and Q is the poset whose underlying set is the disjoint union of P and Q and whose order relation is given by $x < y$ if either (i) $x <_P y$, (ii) $x <_Q y$, or (iii) $x \in P$ and $y \in Q$. Clearly

$$\Delta(P * Q) = \Delta(P) * \Delta(Q).$$

There are topological relationships between the join and product of posets, which are discussed in Section 5.1.

1.2. The Möbius function

The story of poset topology begins with the Möbius function $\mu(= \mu_P)$ of a poset P defined recursively on closed intervals of P as follows:

$$\begin{aligned} \mu(x,x) &= 1, \quad \text{for all } x \in P \\ \mu(x,y) &= -\sum_{x \le z < y} \mu(x,z), \quad \text{for all } x < y \in P. \end{aligned}$$

For a bounded poset P, define the Möbius invariant

$$\mu(P) := \mu_P(\hat{0}, \hat{1}).$$

In Figure 1.2.1, the values of $\mu(\hat{0}, x)$ are shown for each element x of the poset.

There are various techniques for computing the Möbius function of a poset; see [**169**]. Perhaps the most basic technique is given by the product formula.

Proposition 1.2.1. *Let P and Q be posets. Then for $(p_1, q_1) \le (p_2, q_2) \in P \times Q$,*

$$\mu_{P \times Q}((p_1, q_1), (p_2, q_2)) = \mu_P(p_1, p_2)\mu_Q(q_1, q_2).$$

Exercise 1.2.2. Prove Proposition 1.2.1.

Exercise 1.2.3. Use the product formula to show that the Möbius function for the subset lattice B_n is given by

$$\mu(X, Y) = (-1)^{|Y - X|} \quad \text{for all } X \subseteq Y.$$

Exercise 1.2.4. For positive integer n, the lattice D_n of divisors of n is the set of positive divisors of n ordered by $a \leq b$ if a divides b. Show that the Möbius function for D_n is given by

$$\mu(d, m) = \mu(m/d),$$

where $\mu(\cdot)$ is the classical Möbius function of number theory, which is defined on the set of positive integers by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ is divisible by a square.} \end{cases}$$

This example is the reason for the name *Möbius function of a poset*.

The combinatorial significance of the Möbius function was first demonstrated by Rota in 1964 in his Steele-prize winning paper [**147**]. The Möbius function of a poset is used in enumerative combinatorics to obtain inversion formulas.

Proposition 1.2.5 (Möbius inversion)**.** *Let P be a poset and let $f, g : P \to \mathbb{C}$. Then*

$$g(y) = \sum_{x \leq y} f(x)$$

if and only if

$$f(y) = \sum_{x \leq y} \mu(x, y)\, g(x).$$

Three examples of Möbius inversion are classical Möbius inversion ($P = D_n$), inclusion-exclusion ($P = B_n$), and Gaussian inversion ($P = B_n(q)$, the lattice of subspaces of an n-dimensional vector space over the field with q elements); see [**169**] for details.

Our interest in the Möbius function stems from its connection to the Euler characteristic. The reduced Euler characteristic $\tilde{\chi}(\Delta)$ of a simplicial complex Δ is defined to be

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i\, f_i(\Delta),$$

where $f_i(\Delta)$ is the number of i-faces of Δ.

Proposition 1.2.6 (Philip Hall Theorem)**.** *For any poset P,*

$$\mu(\hat{P}) = \tilde{\chi}(\Delta(P)).$$

Exercise 1.2.7. Prove Proposition 1.2.6.

It follows from the Euler-Poincaré formula below that the Euler characteristic is a topological invariant. Hence by Proposition 1.2.6, $\mu_P(x, y)$ depends only on the topology of the open interval (x, y) of P.

Theorem 1.2.8 (Euler-Poincaré formula)**.** *For any simplicial complex Δ,*

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i\, \tilde{\beta}_i(\Delta),$$

where $\tilde{\beta}_i(\Delta)$ is the ith reduced Betti number of Δ, i.e., the rank, as an abelian group, of the ith reduced homology of Δ over $\mathbb{Z}$.