APRIL 7

Contents

1. Fan defined by a Polytope

Let $P \subset M_{\mathbb{R}^n} = M_{\mathbb{R}}$ be an n-dimensional polytope such that for every vertex m, the semigroup $\mathbb{N}(P \cap M - m)$ is saturated. We have seen that it defines a projective toric variety $X_{P\cap M} = X_{\mathcal{A}}$.

$$X_{P\cap M} = \bigcup_{m\,vertex} Spec(\mathbb{C}[\check{\sigma_m} \cup M])$$

where $\check{\sigma_m} = Cone(P \cap M - m)$.

Because $\dim(P) = n$ we can write:

$$P = \bigcap_{F \subset P \text{ facet}} H_{n_F,a_F}.$$

Let $m \in P$ be a vertex. By duality one sees that

$$\sigma_m = Cone(n_F \text{ s.t. } m \in F).$$

Recalling that every proper face, $Q \subset P$, is the intersection of supporting hyperplanes and that the intersection of a suporting hyperplane with polytope determines a face, we see that:

$$Q = \bigcap_{Q \subseteq F} F.$$

Therefore every face defines a cone

$$\sigma_Q = Cone(n_F \text{ s.t. } Q \subset F),\, \sigma_Q \subset \sigma_m \text{ for every vertex } m \in Q$$

Observe also that $\dim(Q) = \dim(\check{\sigma_Q})$ because $\check{\sigma_Q} = < n_{F_1}, \check{\ldots}, n_{F_s} >$ where $Q = F_1 \cap \ldots \cap F_s$.

Then every face $Q \subseteq P$ defines a cone $\sigma_Q = Cone(Q \check{\cap} M - m) \subset Cone(Q \check{\cap} M - m) = \sigma_m$ form $m \in \sigma$ such that

$$\dim(Q) + \dim(\sigma_Q) = n.$$

For every facet F the cone σ_F is one dimensional, and it is one of the edged of σ_m for every $m \in F$. The Polytope defines $\sigma_P = Cone() = \{0\}$.

Lemma 1.1. *Let $Q \subseteq P$ be a face and let $H_{u,b}$ be a supporting hyperplane. Then*

$$u \in \sigma_Q \Leftrightarrow Q \subseteq H_{u,b} \cap P$$

Proof. Let $Q = F_1 \cap ... \cap F_s$ so that $\sigma = Cone(u_{F_1}, ..., u_{F_s}$. Notice that a supporting hyperplane is of the form $H_{u,b}$, is such that $Q \subset H_{u,b} \cap P$ if $< m, u >= b$ for every $m \in Q = F_1 \cap ... \cap F_s$ (and $< m, n_{F_i} >= -a_i$.)

If $u \in \sigma_Q$, then $u = \sum_1^s \lambda_{F_i} u_{F_i}$. Then $b = \sum_1^s \lambda_i a_i$ is such that $Q \subset H_{u,b} \cap P$.

If $Q \subset H_{u,b} \cap P$ consider a vertex $m \in Q$. Then $m \in H_{u,b}$ and $P \subset H^+_{u,b}$. It follows that $P \cap M - m \subset H^+_{u,0}$. Let now $u \in \sigma_m$ and write $u = \lambda_F u_F$ as before. If Q is a facet, then $H_{u_F,a_F} \cap P = F$ and $Cone(u_F) = \sigma_F$. We may assume that Q is not a facet. Let $m \in F_1$ and let $p \in Q$ and $p \in (Q \setminus F_1)$. Then

$$< p, u_{F_i} > \geq -a_{F_i}, < p, u_{F_1} >> -a_{F_1}$$

Moreover

$$< p, u >= \sum -\lambda_{F_i} < p, u_{F_i} >= b, < m, u >= \sum -\lambda_{F_i} a_{F_i} = b$$

gives $\sum -\lambda_{F_i} < p, u_{F_i} >= \sum -\lambda_{F_i} a_{F_i}$, which implies $\lambda_{F_1} = 0$. This for every face such that $Q \not\subset F$. $\square$

As a corollary we have that

$$u_F \in \sigma_Q \Leftrightarrow Q \subseteq F.$$

Lemma 1.2. *Lat Q, Q' be faces of P, then*

(1) $Q \subseteq Q' \Leftrightarrow \sigma_{Q'} \subseteq \sigma_Q$.
(2) $\sigma_Q \cap \sigma_{Q'} = \sigma_{Q''}$, where Q'' is the smallest face containing Q and Q'.

Proof. (1) If $Q \subset Q'$ if and only if $Q \subseteq F$ for every $Q' \subset F$, which
(2) Let Q'' is the smallest face containing Q and Q'. One sees that

$$Q'' = \bigcap_{Q \subseteq F, Q' \subseteq F} F,$$

Because then $\sigma_{Q''} = Cone(n_F, Q \subseteq F, Q' \subseteq F)$ it follows that $\sigma_{Q''}$ is a face of both σ_Q and $\sigma_{Q'}$ and thus $\sigma_{Q''} \subset \sigma_Q \cap \sigma_{Q'}$.

If $\sigma_{Q'} \subset \sigma_{Q''} = \sigma_P = \{0\}$, then $Q'' = P$. Otherwise for $u \in \sigma_{Q'} \subset \sigma_{Q''}$ we have $Q \subseteq H_{u,b} \cap P, Q' \subseteq H_{u,b} \cap P$ for some $b \in \mathbb{R}$. Because $H_{u,b} \cap P$ is a face it is $Q'' \subseteq H_{u,b} \cap P$ for the minimality of Q''. Then again Lemma implies that $u \in \sigma_{Q''}$ and thus $\sigma_{Q''} \subset \sigma_Q \cap \sigma_{Q'}$.

$\square$

2. The inner-normal fan

We have seen that, if we collect all cones defined by faces:

$$\Sigma_P = \{\sigma_Q \text{ s.t. } Q \text{ is a face of } P\},$$

this collection has the following properties:

- For every $\sigma_Q \in \Sigma_P$ faces of σ_Q belong to Σ_P.
- Intersection of cone in Σ_P belong to Σ_P.

A collection of lattice cones with the above properties id called a *fan*. Because the cones in Σ_P are generated by the inner-normal vectors of facets, the fan Σ_P is called The *inner-normal fan*.

Moreover it is:

$$\bigcup_{m \text{ vertex of } P} \sigma_m = N_{\mathbb{R}}.$$

In fact if $0 \neq u \in N_{\mathbb{R}}$, let $b = min\{< u, m >, m \text{ vertex of } P\}$. Then $P \subset H^+_{u,b}$ and $m \in H_{u,b}$ for at least one vertex m. Lemma 2 implies that $u \in \sigma_m$.

A fan whose maximal dimensional cones cover the whole space is called *complete*.

Example 2.1. P = pentagon in $\mathbb{R}^2$. See [CLS, Example 3.10] for a nice example in $\mathbb{R}^3$.

3. Correspondence faces-affne open subspaces

Consider $\tau \subseteq \sigma$, i.e. $\tau = \sigma \cap H_m$ for $m \in \check{\sigma} \cap M$.

Then $S_\sigma \subseteq S_\tau$ and $\pm m \in \tau^\perp$, since $< m, u >= 0$ for every $u \in \tau$. It follows that $S_\sigma + \mathbb{Z}(-m) \subset S_\tau$, and that

$$\mathbb{C}[S_\tau] = \mathbb{C}[S_\sigma]_{\chi^m}.$$

This implies that for each face $\tau \subset \sigma$, $Spec(\mathbb{C}[S_\tau])$ is a Zariski open subset of $Spec(\mathbb{C}[S_\sigma])$

In particular if m, m' are verteces pf P, $\tau\sigma_m \cap \sigma_{m'}$ is a face of both σ_m and $\sigma_{m'}$. Moreover one sees that $\tau = \sigma_m \cap H_{m'-m}$, which implies:

$$X_m \cap X_{m'} = Spec(\mathbb{C}[S_m]_{\chi^{m'-m}}.)$$

As already observed.

Notice that even if the a polytope P is not saturated, i.e., there is some S_{m_i} which is not saturated, there is a (possibly big) positive integer k such that kP is saturated. Then for any maximal dimensional polytope P we can define the associated projective toric variety as

$$X_{kP}, \text{ where } k \text{ is any positive integer such that } kP \text{ is saturated .}$$

It is well defined since:

Exercise 3.1. HMW II Let $P = \cap_1^d H^+_{n_{F_i},a_i} \subset \mathbb{R}^n \cong M_\mathbb{R}$ be a maximal dimensional lattice polytope, with facets $F_1, ..., F_d$. For any $k \geq 1$ we denote by kP the polytope:

$$kP = \cap_1^d H^+_{n_{F_i},ka_i}.$$

If $m \in M$, then $P + m$ is the polytope translated by the translation $x \mapsto x + m$. Then for any $m \in M$ and any integer $k \geq 1$,

$$\Sigma_P = \Sigma_{P+m} = \Sigma_{kP},$$

and thus they define the same projective toric variety. Polytopes defining the same projective toric variety are said to be combinatorially equivalent, or to have the same combinatorial type.

Example 3.2. $P_{a,b} = Conv((0,0,(a,0),(0,1),(b,1))$ where $1 \leq a \leq b$. The fan only depend on $b - a$.

$$X_{P_{a,b}\cap M} = \mathbb{F}_{b-a}.$$

is called the Hirzebruch surface of degree $r = b - a$.

References

[AM] Atiyah, M. F.; Macdonald, I. G. Introduction to commutative algebra. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969 ix+128 pp.
[CLS] D. Cox, J. Little, H. Scenck. Toric Varieties.
[F] W. Fulton.Introduction to toric varieties.Annals of Math. Princeton Univ. Press, 131.