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Contents

1. Affine varieties

Lemma 1.1. *Let V, W be affine varieties*

(1) $V \subseteq W \Leftrightarrow I(W) \subseteq I(V)$.
(2) $V = W \Leftrightarrow I(W) = I(V)$.
(3) $I(V)$ is prime $\Leftrightarrow V$ is irreducible;
(4) (Weak Nullstellensatz) For $x \in V$, then $I(\{x\}) := m_x$ is a maximal ideal. Moreover all the maximal ideal of $\mathbb{C}[x_1, ..., x_n]$ are of the form $m_x = ((x_1 - a_1, ..., x_n - a_n)$. This means that there is a one to one correspondence:

$$\{ \text{ maximal ideals of } \mathbb{C}[x_1, ..., x_n]\} \leftrightarrow \{ \text{ points of } \mathbb{C}^n\}.$$

Proof. (1) Easy.

(2) Easy.

(3) Assume $I(V)$ is prime and let $V = V_1 \cup V_2$. Then $I(V) \subset I(V_1)$ and thus we can take $f \in I(V_1) \setminus I(V)$. For any $g \in I(V_2)$ the product $fg \in I(V)$ and thus because $I(V)$ is prime and $f \notin I(V)$ it is $g \in I(V)$. This imples that $I(V) = I(V_2)$ and thus $V_2 = V$.

Assume now that V is irreducible and that $fg \in I(V)$. Let $V_1 = V(f) \cap V, V_2 = V(g) \cap V$. It follows that $V = V_1 \cup V_2$ and that $V = V_1$ (i.e. $f \in I(V)$) or $V = V_2$ (i.e. $g \in I(V)$).

(4) Let $x = (a_1, ..., a_n)$, and consider $I = (x_1 - a_1, ..., x_n - a_n)$. We first prove that every ideal of the form $I = (x_1 - a_1, ..., x_n - a_n)$ is maximal. Assume $I \subset M \subset R$. then there is $1 \neq f \in M \setminus I$ and by the division algorithm write:

$$f = A_1(x_1 - a_1) + ... + A_n(x_n - a_n) + b,$$

where $0 \neq b \in \mathbb{C}$. Moreover, since $A_1(X_1 - a_1) + ... + A_n(x_n - a_n) \in I \subset M$ and $f \in M$, it follows that $b \in M$ and hence $1 = b \cdot \frac{1}{b} \in M$, i.e. $M = R$. Now we prove that every maximal ideal in $\mathbb{C}[x_1, ..., x_n]$ is of the form $(x_1 - a_1, ..., x_n - a_n)$. Moreover, since $(x_1 - a_1, ..., x_n - a_n) \subset m_x$, it has to be $(x_1 - a_1, ..., x_n - a_n) = m_x$. Let $I \subset \mathbb{C}[x_1, ..., x_n]$ be maximal. Since I is proper it is $I \neq \mathbb{C}[x_1, ..., x_n]$ and thus $V(I) \neq \emptyset$. Let $x = (a_1, ..., a_n) \in V(I)$. It follows that for every $f \in I$, $f \in m_x$ and thus $I \subset m_x$. It follows that $I = (x_1 - a_1, ..., x_n - a_n) = m_x$. $\square$

We have established a correspondence:

$$\text{affine varieties} \ \to^I \ \text{ideals}$$

$$\text{ideals} \ \to^V \ \text{affine varieties}$$

This maps are inclusion-reversing and $V(I(X)) = X$.

THEOREM 1.2. *(Hilbert Nullstellensatz)* $I(V(J)) = \sqrt{J}$.

Proof. (Outline) Let $J = V(f_1, ..., f_k)$ and $f \in I(V(J))$. We have to show that there is $m \geq 1$ and $A_i \in \mathbb{C}[x_1, ..., x_n]$ such that $f^m = \sum A_i f_i$. Consider $J' = (f_1, ..., f_k, 1 - yf) \subset \mathbb{C}[x_1, ..., x_n, y]$ and let $(a_1, ..., a_n, a_{n+1}) \in \mathbb{C}^{n+1}$. If $\underline{a} = (a_1, ..., a_n) \in V(J)$, then $f(\underline{a}) = 0$ and $1 - a_{n+1} f(\underline{a}) = 1 \neq 0$ which implies that $(a_1, ..., a_n, a_{n+1}) \notin V(J')$.

If $\underline{a} = (a_1, ..., a_n) \notin V(J)$ then $f_i(\underline{a}) \neq 0$ for some i. This means that $f_i(a_1, ..., a_n, a_{n+1}) \neq 0$ for some i and thus again $(a_1, ..., a_n, a_{n+1}) \notin V(J')$.

We can conclude that $V(J') = \emptyset$. (By induction on n) one proves that if $V(J') = \emptyset$, then $J' = \mathbb{C}[x_1, ..., x_n, y]$, that is

$$1 = \sum p_i(x_1, ..., x_n, y) f_i + q(x_1, ..., x_n, y)(1 - yf).$$

Substituting $y = f^{-1}$ we get $1 = \sum p_i(x_1, ..., x_n, f^{-1}) f_i$ and for m sufficiently large to clear denominators:

$$f^m = \sum A_i f_i.$$

The other inclusion is obvious. $\square$

Example 1.3. $I(V_{\mathbb{R}}(y - x^2, z - x^3)) = (y - x^2, z - x^3)$.

There is then a bijection:

$$\text{Radical ideals} \ \to^V \ \text{affine varieties}$$

(an ideal I is radical if $I = \sqrt{I}$.)

2. Isomorphic affine varieties

A *polynomial function* on an affine variety $V \subset \mathbb{C}^n$ is the restriction of a polynomial on $\mathbb{C}^n$.

The ring $\mathbb{C}[V] := \mathbb{C}[x_1, ..., x_n]/I(V)$ is called the *coordinate ring* of the variety V.

Because f, g restricts to the same function on V if and only if $f - g \in I(V)$ we have that:

$$\mathbb{C}[V] = \{f : V \to \mathbb{C} \text{ s.t. } f \text{ is a polynomial function}\}.$$

From what said above we see
$\mathbb{C}[V]$ is an integral domain $\Leftrightarrow$ $I(V)$ is prime $\Leftrightarrow$ V is irreducible;

Moreover, because the maximal ideals in $\mathbb{C}[V]$ correspond to the maximal ideals m_x in $\mathbb{C}[x_1, ..., x_n]$ such that $I(V) \subset m_x$, it follows that there is a one to one correspondence:

$$\{ \text{ maximal ideals of } \mathbb{C}[V]\} \leftrightarrow \{ \text{ points of } V\}.$$

Definition 2.1. Let $V \subset \mathbb{C}^n, W \subset \mathbb{C}^m$ be two affine varieties.

A map $f : V \to W$ is a polynomial map if there exist m polynomials $F_1, ..., F_M \in \mathbb{C}[x_1, ..., x_n]$ such that:

$$f(x) = (F_1(x), ..., F_m(x)).$$

Equivalently f is a polynomial map iff $y_j \circ f \in \mathbb{C}[V]$ for $j = 1, ..., m$, where y_i are the coordinate functions on W.

Definition 2.2. A polynomial map $f : V \to W$ is an isomorphism (of affine varieties) if there is a polynomial map $g : W \to V$ such that $f \circ g = id_W, g \circ f = id_V$.

Example 2.3. Let $C = V(y - x^2, z - x^3) \subset \mathbb{C}^3$, the map:

$$f : \mathbb{C} \to C, f(t) = (t, t^2, t^3)$$

is a polynomial map and an isomorphism.

Every polynomial map $f : V \to W$ induces a ring homomorphism:

$$f^* : \mathbb{C}[W] \to \mathbb{C}[V]$$

define by $f^*(g) = g \circ f$. Conversely every algebra (hence ring) homomorphism $F : \mathbb{C}[W] \to \mathbb{C}[V]$ is of the form $F = f^*$ for some $f : V \to W$. In other words there is a bijection:

$$\{ \text{ polynomial maps } f : V \to W\} \leftarrow \{\text{algebra homomorphism } F : \mathbb{C}[W] \to \mathbb{C}[V]\}$$

such that $(g \circ f)^* = f^* \circ g^*$. We can conclude that: $f : V \to W$ is an isomorphism iff $f^* : \mathbb{C}[W] \to \mathbb{C}[V]$ is an isomorphism.

We see hence that:

Example 2.4.

$$\mathbb{C}[\mathbb{C}^*] = \mathbb{C}[x, x^{-1}] \text{ Laurent polynomials in one variable .}$$

Similarly:

$$\mathbb{C}((\mathbb{C}^*)^n) = \mathbb{C}[x_1, x_1^{-1}, ..., x_n, x_n^{-1}].$$

Because $(\mathbb{C}^*)^n = \mathbb{C}^* \times ... \times \mathbb{C}^* \cong V(x_1y_1 - 1, ..., x_ny_n - 1) \subset \mathbb{C}^{2n}$. Observe that $\mathbb{C}((\mathbb{C}^*)^n)$ is an Integral Domain (it a localization of an Integral domain!) and hence the affine variety $(\mathbb{C}^*)^n$ is irreducible.

In what follows we will assume that the affine variety V is irreducible. The field of fractions

$$\mathbb{C}(V) = \{\frac{f}{g} \text{ s.t. } f, g \in \mathbb{C}[V], g \neq 0\}$$

is called the *function field* or the field of *rational functions* on V.

Definition 2.5. A function $f \in \mathbb{C}(V)$ is *regular* at $p \in V$ if f can be represented as as $f = \frac{h}{g}$, where $g(P) \neq 0$.

Notice that, since $\mathbb{C}(V)$ will not be in general a U.F.D. en element f can have more than one representation. Moreover Let $Dom(f) = \{p \in V | f \text{ is regular at } P\}$

Lemma 2.6.

$$\textit{a rational function } f \textit{ is regular on the whole } V \Leftrightarrow f \in \mathbb{C}[V].$$

Proof. Let $D_f = \{g \in \mathbb{C}[V] \text{ s.t. } fg \in \mathbb{C}[V]\}$. It is an ideal called theideal of the denominators. Moreover it is $V \setminus Dom(f) = V(D_f)$. It follows that:

$$Dom(f) = V \Leftrightarrow V(D_f) = \emptyset \Leftrightarrow D_f = \mathbb{C}[V] \Leftrightarrow 1 \in D_f \Leftrightarrow f \in \mathbb{C}[V]$$

□

Definition 2.7. Let V be an affine variety. A *rational map*:

$$f : V - - > \mathbb{C}^m$$

is a partially defined map given by rational functions $f_1, ..., f_m \in \mathbb{C}(V)$, i.e.

$$f(p) = (f_i(p), ..., f_m(p)), p \in \cap Dom(f_i).$$

Moreover f is regular at p if $p \in \cap Dom(f_i)$.

Let $V \subset \mathbb{C}^n, W \subset \mathbb{C}^m$. A rational map $f : V - - > W$ is a rational map $f : V - - > \mathbb{C}^m$ such that $f(Dom(f)) = W$.

Example 2.8. Consider $f(x, y, z) = (x, y, z, \frac{xy}{z})$ is a rational map over $(\mathbb{C}^*)^3$. Notice that the inverse of this map id the projection onto the first three factors which is regular. Moreover $Dom(f) \subset V(xy - zw)$, in fact $Dom(f) = V(xy - zw) \setminus V(z)$. This is an example of an affine variety that contains a Zariski open "isomorphic" to a product of $\mathbb{C}^*$.

3. Zariski-open subspaces

Definition 3.1. The complement of an affine variety $V(f_1, ..., f_k) \in \mathbb{C}^n$

$$\mathbb{C}^n \setminus V(f_1, ..., f_k) = \{x | f_i(x) \neq 0 \text{ for some } i = 1, ..., k\}$$

is called a *Zariski open set*.

Now we will make precise what do we mean by isomorphism of open subsets:

Definition 3.2. Let V, W be irreducible affine varieties and let $U \subset V$ be a Zariski open subset. A morphism $f : U \to W$ is a *rational* map if $f : V - - > W$ such that $U \subset Dom(f)$.

If $U_1 \subset V$ and $U_2 \subset W$ are open then a morphism $f : U_1 \to U_2$ is a morphism $f : U_1 \to W$ such that $f(U_1) = U_2$.

An *isomorphism* is a morphism which has an inverse which is a morphism.

Note that

$$\mathbb{C}^n \setminus V(f_1, ..., f_k) = \mathbb{C}^n \setminus (V(f_1) \cap ... \cap V(f_k)) = (\mathbb{C}^n \setminus V(f_1)) \cap ... \cap (\mathbb{C}^n \setminus V(f_1))$$

We will denote these building blocks $(\mathbb{C}^n \setminus V(f_i)) = V_{f_i}$.

Let $I = I(V) \subset \mathbb{C}[x_1, ..., x_n]$, $f \in \mathbb{C}[V]$ and $V_f = V - V(f)$. Clearly we can interpret V_f as an affine variety. Let $J = (I, yf - 1)$ and $V(J) \subset \mathbb{C}^{n+1}$. The map $f(x) = (x, \frac{1}{f(x)})$ is a morphism from V_f to $\mathbb{C}^{n+1}$ with inverse $f^{-1}(x, y) = x$ which is regular. it follows that

$$V_f \cong V(J)$$

Notice that

$$\mathbb{C}[V_f] = \mathbb{C}[V][f^{-1}] = \{\frac{g}{f^l} \in \mathbb{C}(V) \text{ s.f. } g \in \mathbb{C}[V]\}.$$