(j) Not all Boolean algebras are isomorphic to an algebra of all subsets of a set. (Show by example that there are countable Boolean algebras.)

Note This investigation is completed in 5.S.

L FILTERS

A theory of convergence has been built on the concept of filter. A *filter* $\mathfrak{F}$ in a set X is a family of non-void subsets of X such that

(i) the intersection of two members of $\mathfrak{F}$ always belongs to $\mathfrak{F}$; and
(ii) if $A \in \mathfrak{F}$ and $A \subset B \subset X$, then $B \in \mathfrak{F}$.

In the terminology of the previous problem a filter is a proper dual ideal in the Boolean ring of all subsets of X. A filter $\mathfrak{F}$ *converges* to a point x in a topological space X iff each neighborhood of x is a member of $\mathfrak{F}$ (that is, the neighborhood system of x is a subfamily of $\mathfrak{F}$).

(a) A subset U is open iff U belongs to every filter which converges to a point of U.

(b) A point x is an accumulation point of a set A iff $A \sim \{x\}$ belongs to some filter which converges to x.

(c) Let ϕ_x be the collection of all filters which converge to a point x. Then $\bigcap\{\mathfrak{F}: \mathfrak{F} \in \phi_x\}$ is the neighborhood system of x.

(d) If $\mathfrak{F}$ is a filter converging to x and $\mathfrak{G}$ is a filter which contains $\mathfrak{F}$, then $\mathfrak{G}$ converges to x.

(e) A filter in X is an *ultrafilter* iff it is properly contained in no filter in X. If $\mathfrak{F}$ is an ultrafilter in X and the union of two sets is a member of $\mathfrak{F}$, then one of the two sets belongs to $\mathfrak{F}$. In particular, if A is a subset of X, then either A or $X \sim A$ belongs to $\mathfrak{F}$. (Problem 2.I again.)

(f) One might suspect that filters and nets lead to essentially equivalent theories. Grounds for this suspicion may be found in the following facts:

(i) If $\{x_n, n \in D\}$ is a net in X, then the family $\mathfrak{F}$ of all sets A such that $\{x_n, n \in D\}$ is eventually in A is a filter in X.
(ii) Let $\mathfrak{F}$ be a filter in X and let D be the set of all pairs (x,F) such that $x \in F$ and $F \in \mathfrak{F}$. Direct D by agreeing that $(y,G) \geq (x,F)$ iff $G \subset F$, and let $f(x,F) = x$. Then $\mathfrak{F}$ is precisely the family of all sets A such that the net $\{f(x,F), (x,F) \in D\}$ is eventually in A.

Notes The definition of filter is due to H. Cartan; his treatment of convergence is given in full in Bourbaki [1]. Proposition (c) is a remark of W. H. Gottschalk; (f) is part of the folklore of the subject.