Homework, Topology, fall 1998. Version December 1.

Here are 25 homework problems. Correct and well-written solutions of at least one half of these will be enough for the mark "godkänd" or "3". For higher marks more will be needed (more solutions or possibly passing some other kind of examination).

1. Find all the limit points of the following subsets of $\mathbf{R}$.

a) $\{1/m + 1/n : m, n = 1, 2, ...\}$.

b) $\{\frac{\sin n}{n} : n = 1, 2, ...\}$.

2. Let A be dense and U open in a topological space. Show that $U \subset clos(A \cap U)$.

3. Show that every closed set in $\mathbf{R}^2$ is the boundary of some subset of $\mathbf{R}^2$.

4. The following family of sets forms a basis of a topology τ on R: the sets $(x-\varepsilon, x+\varepsilon)$ for all $x \in \mathbf{R} \setminus \{0\}$ and all ε with $0 < \varepsilon < |x|$ together with the sets $(-\varepsilon, \varepsilon) \cup (-\infty, -n) \cup (n, \infty)$ for all $\varepsilon > 0$, $n \in \mathbf{N}$.

Draw a picture of $(\mathbf{R}, \tau)$ embedded in $\mathbf{R}^2$ in such a way that τ corresponds to the subspace topology of $\mathbf{R}^2$.

5. a) Show that the family of all half-open intervals $[a, b)$ (where $a, b \in \mathbf{R}$) forms a basis of a topology τ on $\mathbf{R}$.

b) Show that the rational numbers are dense in $(\mathbf{R}, \tau)$.

c) Does there exist a countable basis for τ?

d) Let σ denote the usual topology of $\mathbf{R}$. Set $f(x) = \sin x$. Is f continuous considered as a function $f : (\mathbf{R}, \sigma) \to (\mathbf{R}, \tau)$?

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6. a) Suppose a topological space X has a countable basis. Show that X is separable.

b) Does the converse implication (from separable to the existence of a countable basis) hold? (Hint: problem 5.)

c) Show that the converse (in b)) holds in case X is a metric space.

7. Let A be a subset of a topological space X. Show that at most 14 different sets (including A itself) can be obtained from A through the operations of taking closure and taking complement with respect to X.

Find a subset A of $\mathbf{R}$ (with its usual topology) so that the number 14 above really is attained.

8. Construct a topological space X and a compact subset K of it such that the closure of K is not compact.

9. Recall that a **net** in a topological space X is a map $\Lambda \to X$, where Λ is a **directed set**, i.e. a set having a relation $\prec$ such that (i) $\alpha \prec \beta$ and $\beta \prec \gamma$ implies $\alpha \prec \gamma$ and (ii) for every two α, $\beta \in \Lambda$ there exists $\gamma \in \Lambda$ with $\alpha \prec \gamma$ and $\beta \prec \gamma$. Recall also that the net, write it as $\Lambda \ni \alpha \mapsto x_\alpha \in X$, converges to a point $x \in X$ if and only if for every neighbourhood U of x there is a $\gamma \in \Lambda$ such that $x_\alpha \in U$ for all $\alpha \in \Lambda$ with $\gamma \prec \alpha$. We then write $x_\alpha \to x$.

a) Show that a function $f : X \to Y$ is continuous if and only if for every convergent net $x_\alpha \to x$ in X we have $f(x_\alpha) \to f(x)$ (in Y).

b) Give an example showing that the statement in a) is false (in general) if we replace the word "net" by "sequence".

c) Show that a topological space X is Hausdorff if and only if every net in X converges to at most one point. (So if X is not Hausdorff you have to construct a net converging to two points simultaneously.)

10. Show that every compact Hausdorff space is homeomorphic to a closed subset of the product space $[0,1]^A$, for some set A.

11. Show that in a topological space X a sequence (x_n) converges to a point x if and only if every subsequence has in its turn a subsequence which converges to x. (The same statement applies to nets.)

12. Let X be a suitable set of functions $(0,1) \to \mathbf{R}$, e.g. the set of Riemann or Lebesgue integrable functions. Recall that there exists a topology on X such that convergence $f_n \to f$ in this topology is the same as pointwise convergence (i.e. convergence $f_n(x) \to f(x)$ for each individual $x \in (0,1)$). This is the topology gotten by regarding X as a subspace of the product space $\mathbf{R}^{(0,1)}$.

Now, show that there is *no* topology on X which corresponds to convergence almost everywhere (a.e.). By definition, $f_n \to f$ a.e. if there exists a nullset (set of Lebesgue measure zero) N such that $f_n(x) \to f(x)$ for every $x \in (0,1) \setminus N$.

Hint: Construct a sequence f_n such that, for each x, $f_n(x) = 1$ for infinitely many values of n but such that, still, every subsequence of f_n has a subsequence which converges a.e. to zero. Then use problem 11.

13. Set $\mathbf{10} = \{0,1,2,3,4,5,6,7,8,9\}$ provided with the discrete topology and $\mathbf{10}^{\mathbf{N}}$ provided with the product topology ($\mathbf{N} = \{1,2,...\}$). Thus $\mathbf{10}^{\mathbf{N}}$ is a compact topological space. We have the "decimal expansion map" (or rather its inverse)

$$f : \mathbf{10}^{\mathbf{N}} \to [0,1]$$

defined by

$$(a_1, a_2, ...) \mapsto 0, a_1 a_2 ... = \sum_{n=1}^{\infty} a_n 10^{-n}.$$

Clearly f is surjective (each number has a decimal expansion), but not quite injective (because certain numbers have two decimal expansions, e.g. $0,300000... = 0,299999....$).

a) Show that f is continuous.

If f were injective it would follow that f were a homeomorphism. We now restrict f to a smaller space so that it becomes injective. Take e.g. $A \subset \mathbf{10}^{\mathbf{N}}$ to consist of those sequences which do not end like 99999..... Then

$$f|_A : A \to [0,1)$$

is injective, and surjective, and $f|_A$ is of course still continuous (with the subset topology on A).

b) Is A closed in $\mathbf{10}^{\mathbf{N}}$? open? dense?.

c) Is the inverse $(f|_A)^{-1} : [0,1) \to A$ continuous? If so, $f|_A$ is a homoemorphism and A and $[0,1)$ are homeomorphic. Is this reasonable?

14. We define a topological space X, the **sheaf of germs of analytic functions** as follows. X consists of all pairs

$$p_0 = (z_0, \{a_n\}_{n=0}^{\infty})$$

where $z_0 \in \mathbf{C}$, $a_n \in \mathbf{C}$ and

$$\limsup(|a_n|)^{1/n} < \infty.$$

The latter means that the a_n are the coefficients of a convergent power series.

With each pair p_0 as above we associate the analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

defined in the open disc $B(z_0, r)$, where $1/r = \limsup(|a_n|)^{1/n}$. For each $z_1 \in B(z_0, r)$ we may expand f in a power series centered at z_1, with coefficients $b_n = f^{(n)}(z_1)/n!$. We then get a new element in X, namely $p_1 = (z_1, \{b_n\})$. Note that the radius of convergence for this pair will be at least $r - |z_0 - z_1|$, but it may be larger. Now we define the topology of X by saying that a typical basic open neighbourhood of p_0 shall consist of all p_1 as above for z_1 in a corresponding neighbourhood $U \subset B(z_0, r)$ of z_0.

Now X is a large topological space with a natural projection map $X \to \mathbf{C}$, $p_0 \mapsto z_0$ which is a local homeomorphism.

Task: Explain, in terms of analytic functions, what are the components X. In particular, answer the following question: which of the pairs below lie in the same component.

$$p_1 = (1, \{0, 1, -1/2, 1/3, -1/4, ...\})$$

$$p_2 = (1, \{\pi i, 1, -1/2, 1/3, ...\})$$

$$p_3 = (1, \{2\pi i, 1, -1/2, 1/3, ...\})$$

$$p_4 = (-1, \{\pi i, -1, -1/2, -1/3, ...\})$$

15. Recall (Armstrong p. 67) that a surjective continuous map which is either open or closed (or both) is an identification map. Give an example of an identification map which is neither open, nor closed.

16. Let S^2 be the unit sphere in $\mathbf{R}^3$ and define $f : S^2 \to \mathbf{R}^4$ by

$$f(x, y, z) = (x^2 - y^2, xy, xz, yz).$$

Show that f induces an embedding of the projective plane into $\mathbf{R}^4$. Can the projective plane be embedded in $\mathbf{R}^3$?

17. Let S^1 denote the unit circle in the plane. Suppose $f : S^1 \to S^1$ is a map which is not homotopic to the identity map. Prove that $f(x) = -x$ for some point $x \in S^1$.

18. A topological space X is said to have the *fixed-point property* if every continuous map from X to itself has at least one fixed point. Which of the following spaces have the fixed-point property:

a) the 2-sphere;

b) the torus;

c) the open unit disc;

d) the one-point union of two circles;

e) the Hilbert cube (i.e., $[0, 1]^{\mathbf{N}}$ with the product topology)?

19. Let $\mathbf{P}^1(\mathbf{C})$ denote the one-dimensional complex projective space, i.e., the identification space obtained from $\mathbf{C}^2 \setminus \{0\}$ by identifying all points which lie on the same complex line. Show that $\mathbf{P}^1(\mathbf{C})$ is identical, as a topological space, with the one-point compactification $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ of the complex plane. (The complex structure on $\mathbf{C}$ extends in a natural way to the point at infinity, and provided with this structure $\hat{\mathbf{C}}$, or $\mathbf{P}^1(\mathbf{C})$, is called the Riemann sphere.)

20. a) Let $p(z)$ be a polynomial of degree ≥ 1 in the complex variable z. Then p is continuous as a function $\mathbf{C} \to \mathbf{C}$. Show that p extends to a continuous function $\hat{\mathbf{C}} \to \hat{\mathbf{C}}$. (See problem 19 for notation.)

b) Can the exponential function $\exp : \mathbf{C} \to \mathbf{C}$ $(\exp(z) = e^z)$ also be extended this way?

c) Recall from complex analysis that every nonconstant analytic function is an open mapping. In particular, this is true for p, in fact even at the point of infinity, i.e., as a map $p : \hat{\mathbf{C}} \to \hat{\mathbf{C}}$. Show that every open continuous map between two compact, connected Hausdorff spaces is surjective, and conclude that the equation $p(z) = 0$ has at least one solution $z \in \mathbf{C}$ (one form of the fundamental theorem of algebra).

21. Construct a triangulation for Klein's bottle and then compute the Euler characteristic for it.

22. Problem 22, p.169, in Armstrong.

23. Let X be a topological space and $\mathcal{U} = \{U_i\}$ a finite open cover of X. The **nerve** $N(\mathcal{U})$ of $\mathcal{U}$ is, by definition, the set of all subcollections $\{U_{i_0}, ..., U_{i_k}\}$ (for various $k \geq 0$) such that $U_{i_0} \cap ... \cap U_{i_k} \neq \emptyset$.

a) Check that $\mathcal{U}$ is an abstract complex. (By an abstract complex is meant simply a finite set V (of "vertices") and a collection S of nonempty subsets ("simplexes") of V such that $s \in S$ and $s' \subset s$ imply $s' \in S$.) *Remark:* The complex $N(\mathcal{U})$ is a basic ingredient in the construction of the Čech homology and cohomology groups.

b) Now, let K be an ordinary (geometric) complex in some Euclidean space $\mathbf{R}^n$. Then the set of open stars of the vertices of K is a finite open cover of $|K|$. Thus we now have two complexes, the nerve of the open stars (abstract) and K itself (geometric). How do they relate to each other?

c) Let X be a simple topological space, e.g. the one-point union of two circles, choose a finite open cover $\mathcal{U}$ of X and compute the edge group $E(N(\mathcal{U}))$ of the nerve of $\mathcal{U}$. (Clearly, the edge group makes sense also for an abstract complex.) Is it isomorphic to the fundamental group of X? The answer may depend on the choice of cover. For a "good" cover (meaning that all intersections of sets in $\mathcal{U}$ are contractible) the groups are supposed to be isomorphic.

24. Let $0 < \alpha < 2\pi$ and consider the points $z_n = e^{i\alpha n}$ on the unit circle C in the complex plane ($n = 1, 2, ...$). The sequence (z_n) defines a filter $\mathcal{F}$ in a canonical way, namely as follows: $F \in \mathcal{F}$ if and only if there exists m such that $z_n \in F$ for all $n \geq m$.

a) Does $\mathcal{F}$ converge to some point?

b) Since C is compact there must anyway exist a refinement $\mathcal{G}$ of $\mathcal{F}$ (i.e., $\mathcal{F} \subset \mathcal{G}$) which converges to some point. Find such a $\mathcal{G}$, which converges to, say, $1 \in C$.

c) Indeed, there should even be an ultrafilter $\mathcal{G} \supset \mathcal{F}$ which converges. Can you find such a $\mathcal{G}$? (Should be possible if α is a rational multiple of π, otherwise it might be more difficult...)

25. Let $X = \mathbf{2}^A$, where $\mathbf{2} = \{0, 1\}$ has the discrete topology and A is any nonempty set. By Tychonov's theorem the space X, when provided with the product topology, is compact for every choice of A. The aim of this problem is to show (if possible?) that X is *sequentially compact* if and only if A is finite or countable. (Recall that a space being sequentially compact means that every infinite sequence has a convergent subsequence.)

a) Show that if A is at most countable, then X is infact sequentially compact. (Not so difficult.)

b) Show that if A has the cardinality of the continuum (e.g. $A = \mathbf{R}$), then X is not sequentially compact. (Maybe difficult, but not impossible.)

c) Is it true that X is not sequentially compact for every uncountable set A?

Hint: If you can show that there exists an injective function $\mathbf{R} \to A$, then you get reduced to case b). However, to prove existence of such a function in general is not easy. (You may need the Christmas holidays, and much more....)