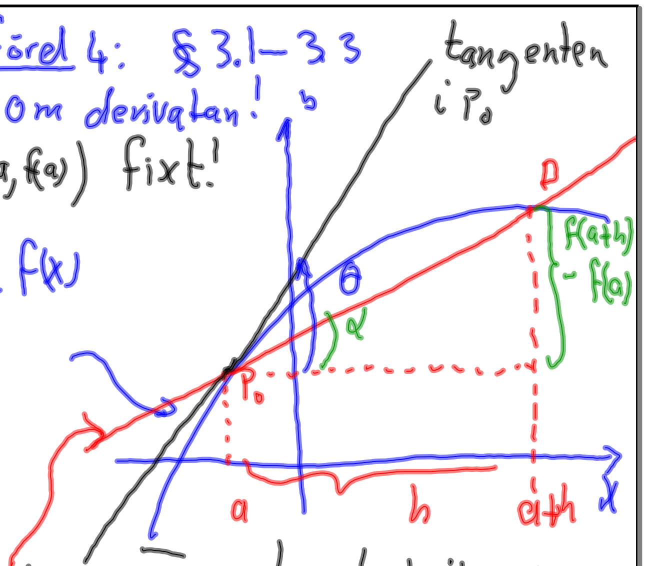
föreläsning 4: § 3.1–3.3
om derivatan!

$P_0=(a,f(a))$ fixt!

$y=f(x)$



kordan $\overline{P_0P}$ har lutningen

$$\tan\alpha = \frac{f(a+h)-f(a)}{h}$$

För $h\to 0$

① $P_0P \longrightarrow$ tangenten!

② $\tan\alpha \longrightarrow \tan\theta$ (Lutning)

$$\therefore \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \text{ETT TAL} = f'(a) = \tan\theta$$

Def 1. f sägs vara deriverbar i $x=a$, om

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = A$$ finns och tecknas $f'(a)$.

olika sätt att skriva derivatan!

$$f'(a) \overset{(*)}{=} \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

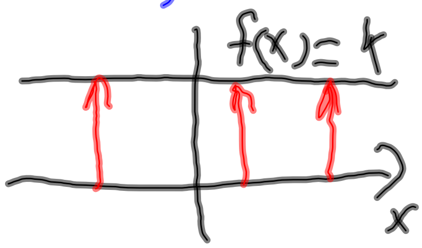
$a+h=x$
$\Leftrightarrow h = x-a$
$h\to 0 \Leftrightarrow x\to a$

$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$a=x$ i $(*)$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Ex1 $f(x)=k \Rightarrow f'(x)=0$



$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{k-k}{h}=0$$

Ex2 $f(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{2\sqrt{x}}, \ x>0$

Ty För $h\neq 0$

$$\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=\left[(a-b)=\frac{a^2-b^2}{a+b}\right]$$

$$=\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

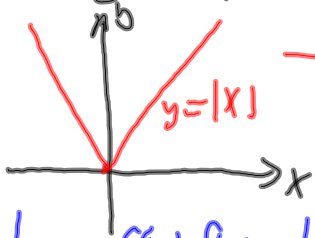
$$=\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

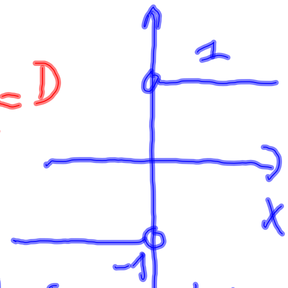
Resultat

$$\boxed{\frac{d}{dx}\left(x^{\alpha}\right)=\alpha\, x^{\alpha-1}}, \ \alpha\in\mathbb{R}$$

Ex3 Kontinuerliga fkner behöver inte vara deriverbara överallt.



$\longrightarrow \frac{d}{dx} = D$



$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|-0}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$$

$$\operatorname{sgn}(x) = \frac{|x|}{x} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$

f ej deriverbar i $x=0$

<u>grundsats</u> sambandet mellan deriverbarhet och kontinuitet

f deriverbar i $x=a \Longrightarrow$ f kont. i $x=a$

(obs! omvänt gäller ej, se föreg. Ex)

Vet f kont i $x=a \iff \lim_{x\to a} f(x) = f(a)$

$$\iff \lim_{x\to a} \big(f(x)-f(a)\big) = 0$$

Vet att f deriverbar i $x=a$

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$f(x)-f(a) = \frac{\big(f(x)-f(a)\big)}{(x-a)}(x-a)$$

$$\lim_{x\to a}\big(f(x)-f(a)\big) = \underbrace{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}_{f'(a)} \cdot \underbrace{\lim_{x\to a}(x-a)}_{=0}$$

$$\therefore \lim_{x\to a}\big(f(x)-f(a)\big) = 0$$

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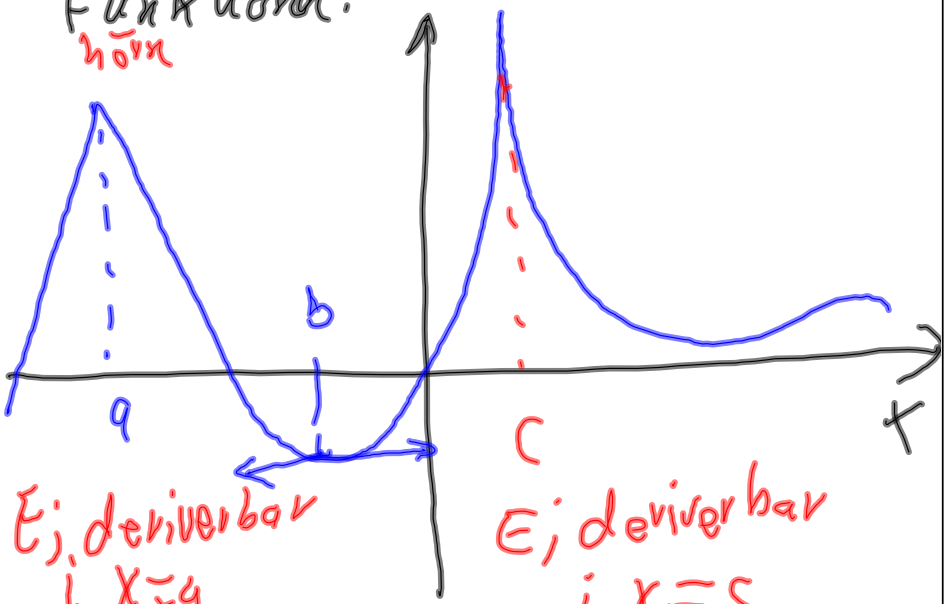
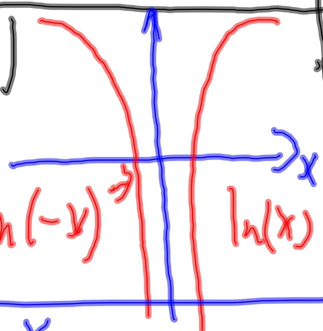
En bild av Deriverbara funktioner.



En liten lista

$f(x)$	$f'(x)$
k = konstant	0
x^α	$\alpha x^{\alpha-1}$ där x är def
$\ln\lvert x\rvert$ 	$\frac{1}{x}$ $x \neq 0$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$1+\tan^2 x = \frac{1}{\cos^2 x}$
$\cot x = \frac{\cos x}{\sin x}$	$-(1+\cot^2 x) = -\frac{1}{\sin^2 x}$

Deriverings regler

1. $\frac{d}{dx}\left(a\, f(x) + b\, g(x)\right) = a\, f'(x) + b\, g'(x)$

$(a, b$ konstanter$)$

2. produktregel

$$\frac{d}{dx}\left(f(x)g(x)\right) = f'(x)g(x) + f(x)g'(x)$$

Ex $\frac{d}{dx}\left(x^{\alpha}\sin x\right) = \left[\begin{array}{l} f(x) = x^{\alpha} \\ g(x) = \sin x \end{array}\right]$

$$= \alpha x^{\alpha-1}\cdot \sin x + x^{\alpha}(\cos x)$$

3. Kvotregeln

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$\frac{d}{dx}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2} \quad \left[\begin{array}{l} \text{här} \\ f(x) = 1 \\ f'(x) = 0 \end{array}\right]$$

Ex $\frac{d}{dx}\left(\frac{1}{\ln|x|}\right) = \left[\begin{array}{l} g(x) = \ln(x) \\ g'(x) = \frac{1}{x} \end{array}\right]$

$$= -\frac{1}{(\ln|x|)^2}\cdot\left(\frac{1}{x}\right)$$

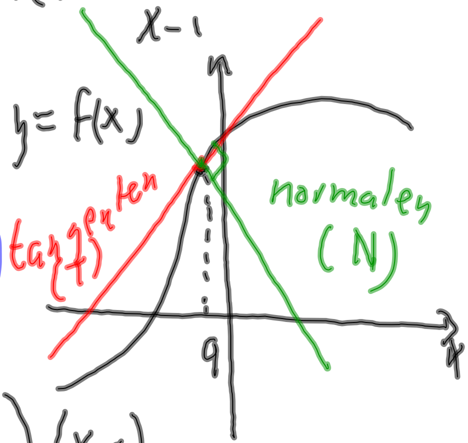
Geomtrisk Tillämpning

Finn ekvationen för tangenten och normalen till $f(x)=\frac{x+1}{x-1}$ i $x=2$

Lösning

ekv. för (T) $y=f(x)$ ges av

$$y=f(a)+f'(a)(x-a)$$



ekv för N

$$y=f(a)+\left(-\frac{1}{f'(a)}\right)(x-a)$$

här $a=2$, $f(2)=\frac{2+1}{2-1}=3$

$$(T):\ y=3+f'(2)(x-2)$$

$$(N):\ y=3+\left(\frac{-1}{f'(2)}\right)(x-2)$$

Finn $f'(2)$

$$f(x)=\frac{x+1}{x-1}=\frac{u}{v}\Rightarrow f'(x)=\frac{u'v-uv'}{v^2}$$

$$=\frac{1.(x-1)-(x+1)1}{(x-1)^2}=-\frac{2}{(x-1)^2}$$

$$\therefore f'(x)=-\frac{2}{(x-1)^2}\Rightarrow f'(2)=\frac{-2}{(2-1)^2}=-2$$

Svar

$$(T):\ y=3+(-2)(x-2)\Leftrightarrow y+2x=7$$

$$(N):\ y=3+\left(\frac{-1}{-2}\right)(x-2)\Leftrightarrow 2y-x=4$$

$$\Leftrightarrow (T):\ y=-2x+7$$

$$(N):\ y=\frac{1}{2}x+2\ (?)$$

Derivatan av Inversen

Låt • f och f^{-1} vara kontinuerliga

• $f'(x)$ finns och $f'(x) \neq 0$

$$\Rightarrow (f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

$y_0 = f(x_0)$

Motivering

$y = f(x) \Leftrightarrow x = f^{-1}(y)$ (1)

$y_0 = f(x_0) \Leftrightarrow x_0 = f^{-1}(y_0)$ (2)

$$\frac{d}{dy} f^{-1}(y) = (f^{-1})'(y_0) = \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$$

$y \neq y_0$ Förenkla!

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = [(1),(2)] = \frac{x - x_0}{f(x) - f(x_0)}$$

$$= \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}.$$

Vidare p.g.a kontinuitet hos f och f^{-1}

$y \to y_0 \Leftrightarrow x \to x_0$

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

olika sätt att skriva $(f^{-1})'$

$y = f(x) \Leftrightarrow x = f^{-1}(y)$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \Leftrightarrow \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$(f^{-1})'(x) = \frac{1}{\frac{d}{dy} f(y)}$$

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Ex $f(x) = (x^2+1)(x+2)$

VET att inversen finns. Finn $(f^{-1})'(0)$?

Lösning Använd.

$$y = f(x) \iff x = f^{-1}(y)$$

$$\frac{dx}{dy} = \frac{d}{dy} f^{-1}(y) = (f^{-1})'(y) = \frac{1}{f'(x)}$$

Använd $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$

då $y_0 = f(x_0)$

Här $y_0 = 0$. Finn x_0:

$$y_0 = f(x_0) \iff 0 = f(x_0)$$

$$\iff 0 = (x_0^2+1)(x_0+2)$$

$$\implies x_0 = -2$$

$$\therefore (f^{-1})'(0) = \frac{1}{f'(-2)}$$

$$f(x) = (x^2+1)(x+2) \Rightarrow f'(x) = 2x(x+2) + (x^2+1)\cdot 1$$

$$\therefore f'(x) = 3x^2+2x+1$$

$$f'(-2) = 3(-2)^2+2(-2)+1 = 9$$

Svar $(f^{-1})'(0) = \frac{1}{9}$

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Tillämpning

$$\boxed{\frac{d}{dx} f^{-1}(x) \overset{*}{=} \frac{1}{\frac{d}{dy} f(y)}}$$

A. $\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}$ $\quad -1<x<1$

Motivering

$y = \arcsin x \iff x = \sin y$ $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $-1 \le x \le 1$

$$\frac{d}{dx}(\arcsin x) \overset{*}{=} \frac{1}{\frac{d}{dy}(\sin y)} = \frac{1}{\cos y}$$

$$= \left[\cos^2 y + \sin^2 y = 1\right] =$$

$$= \frac{1}{\sqrt{1-\underbrace{\sin^2 y}_{x^2}}} = \left[x = \sin y\right] = \frac{1}{\sqrt{1-x^2}}$$

B: $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$ $\quad x \in \mathbb{R}$

Bevis

$y = \arctan x \iff x = \tan y$

$$\frac{dy}{dx} = \frac{d}{dx} \arctan x \overset{(*)}{=} \frac{1}{\frac{d}{dy}(\tan y)}$$

$$\left[\frac{d}{dy} \tan y = 1 + \tan^2 y = \frac{1}{\cos^2 y}\right]$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+\tan^2 y} = \left[x = \tan y\right]$$

$$= \frac{1}{1+x^2}$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C.$$

Kedjeregeln: Att derivera sammansatta funktioner

Låt f vara deriverbar i $u = g(x)$

g — " — i x

bilda $y(x) = f[g(x)]$

$$\frac{dy}{dx} = y'(x) = \frac{d}{dx} f[g(x)] = f'[g(x)] \cdot \underbrace{g'(x)}_{\text{inre deriv}}$$

glöm inte mig

obs!

$$y(x) = f(u(x)) = \frac{df}{du} \frac{du}{dx}$$

Ex $y(x) = \arctan(\sqrt{x}), \quad x>0$

Finn $\frac{d}{dx} \arctan\sqrt{x}$??

Lösning Sätt $u = \sqrt{x}$

$$y(x) = \arctan(u(x)) = f(u(x))$$

$$\frac{dy}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \left[\begin{array}{l} f(u) = \arctan u \\ u(x) = \sqrt{x} \end{array}\right]$$

$$= \underbrace{\frac{d}{du}\arctan u}_{\frac{1}{1+u^2}} \cdot \frac{du}{dx} = \frac{1}{1+u^2} \underbrace{\frac{d}{dx}\sqrt{x}}_{\frac{1}{2\sqrt{x}}}$$

$$= \frac{1}{1+u^2} \cdot \frac{1}{2\sqrt{x}} = [u = \sqrt{x}]$$

$$= \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\left(\arctan\sqrt{x}\right) = \frac{1}{2\sqrt{x}\,(1+x)} \quad x>0$$

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