Förel 7 Om primitiva funktioner

Def 1 F säges vara en primitiv fkn till f om $F'(x) = f(x)$, $x \in$ Interv.

Mängden av alla primitiva fkner tecknas

$$\underbrace{\int f(x)\,dx}_{\text{fkn}} = \underbrace{F(x) + C}_{\text{Fkn}}$$

uttrycket $\int \underbrace{f(x)}_{\text{integranden.}} dx$ kallas för obestämd integral

Def 2

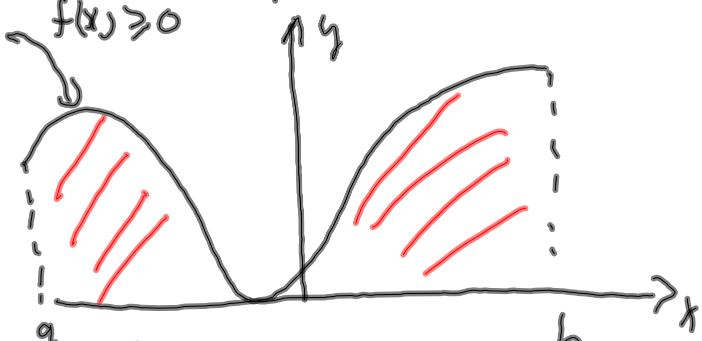
$$\underbrace{\int_a^b f(x)\,dx}_{\text{bestämd integral}} = \text{tal} \gtreqless 0$$

och räknas m h a Insättningsformeln

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Tolkning av $\int_a^b f(x)\,dx = \text{tal}$

$f(x) \geq 0$



$/// = D = \{(x,y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$

$$\int_a^b f(x)\,dx = \text{Area av } D = A(D) \geq 0$$

Om $f(x)$ växlar tecken på $I = [a,b]$

$$A(D) = \int_a^b |f(x)|\,dx \geq 0$$

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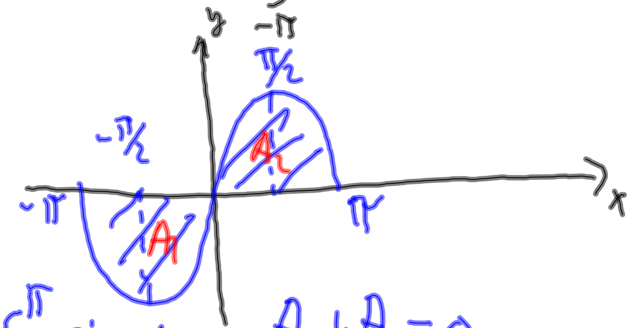
Ex1 Beräkna $\int_{-\pi}^{\pi} \sin x \, dx$

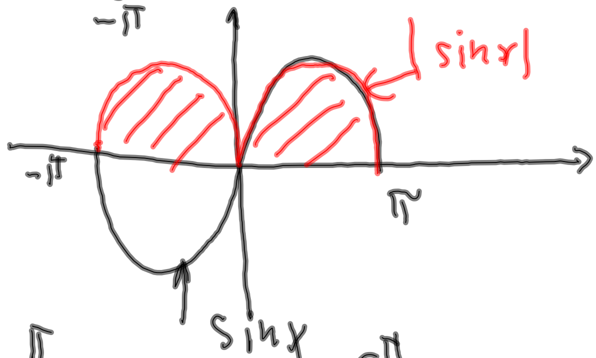


$$\int_{-\pi}^{\pi} \sin x \, dx = A_1 + A_2 = 0$$

$$\int_{-\pi}^{\pi} \sin x \, dx = \left[\begin{array}{l} \frac{d}{dx} \cos x = -\sin x \\ f(x) = \sin x \Rightarrow F(x) = -\cos x \end{array} \right]$$

$$= \Big[F(x)\Big]_{-\pi}^{\pi} = \Big[-\cos x\Big]_{-\pi}^{\pi} = -\underbrace{\Big[\underbrace{\cos \pi}_{=-1} - \underbrace{\cos(-\pi)}_{-1}}_{=0}$$

Ex2 $\int_{-\pi}^{\pi} |\sin x| \, dx$



$$\int_{-\pi}^{\pi} |\sin x| \, dx = 2 \int_{0}^{\pi} \sin x \, dx$$

$$= 2 \Big[-\cos x\Big]_{0}^{\pi}$$

$$= 2 \underbrace{\Big[\underbrace{-\underbrace{\cos(\pi)}_{-1}}_{+1} - \underbrace{(-\underbrace{\cos(0)}_{=1})}_{\substack{-1 \\ +1}}\Big]}_{2 \cdot (1+1) = 4}$$

$$\int_{-\pi}^{\pi} \sin x \, dx = 0$$

Medan $\int_{-\pi}^{\pi} |\sin x| \, dx = 4$

Gör skillnaden

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Elementära primitiva funktioner

$$\int F'(x)\,dx = F(x) + C$$

inre deriv.

kedjeregeln: $$\frac{d}{dx} f[u(x)] = \frac{df}{du}\frac{du}{dx}$$

<u>log.der</u> $$\frac{d}{dx} u(x) = \frac{u'(x)}{u(x)}$$

1. $$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad \alpha \neq -1$$

2. $$\int \frac{1}{x+a}\,dx = \ln|x+a| + C$$

3. $$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C, \quad a \neq 0$$

4. $$\int \cos(ax)\,dx = \frac{\sin(ax)}{a} + C, \quad a \neq 0$$

5. $$\int \sin(ax)\,dx = -\frac{\cos(ax)}{a} + C, \quad a \neq 0$$

6. $$\int \frac{1}{\cos^2 x}\,dx = \int (1+\tan^2 x)\,dx = \tan x + C$$

7. $$\int \frac{1}{\sin^2 x}\,dx = \int (1+\cot^2 x)\,dx = -\cot x + C$$

$$\left(\frac{1}{\sin^2 x} = \frac{\sin^2 x + \cos^2 x}{\sin^2 x} = \frac{\cancel{\sin^2 x}\left(1 + \frac{\cos^2 x}{\sin^2 x}\right)}{\cancel{\sin^2 x}} \right)$$

8. $$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

9. $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

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Metoder att finna primitiva fkner

A. Partiell integration

ur produktregeln för derivatan

$$(uv)' = u'v + uv'$$

$$\Leftrightarrow \quad u'v = (uv)' - uv'$$

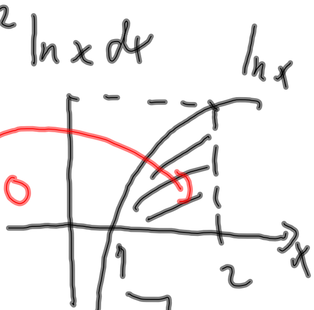
$$\boxed{\int u'v\,dx = uv - \int uv'\,dx}$$

$$\int_a^b u'v\,dx = [uv]_a^b - \int_a^b uv'\,dx$$

Ex: Beräkna $\int_1^2 \ln x\,dx$

Lösning

$$\int_1^2 \ln x\,dx > 0$$



$$\int 1\cdot\ln x\,dx = \begin{bmatrix} u' = 1 \Rightarrow u = x \\ v = \ln x \Rightarrow v' = \frac{1}{x} \end{bmatrix}$$

$$= \int u'v\,dx = uv - \int uv'\,dx$$

$$= x\ln x - \underbrace{\int \cancel{x}\cdot\frac{1}{\cancel{x}}\,dx}_{\int 1\,dx}$$

$$= \underbrace{x\ln x - x}_{F(x)} + C$$

$$\int \ln x\,dx = x\ln x - x + C$$
$$= x(\ln x - 1) + C$$

$$\int_1^2 \ln x\,dx = \left[x(\ln x - 1)\right]_1^2 = 2(\ln 2 - 1) - 1(\underbrace{\ln 1}_{=0} - 1)$$

$$= 2(\ln 2 - 1) + 1$$
$$= 2\ln 2 - 2 + 1 = 2\ln 2 - 1 > 0$$

Svar: $\int_1^2 \ln x\,dx = 2\ln 2 - 1 > 0$

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B. Substitutionen: menas att man gör ett $\boxed{\text{variabelbyte}}$ så att integranden $f(x)$ i $\int f(x)\,dx$ har en "känd" primitiv funktion

Def3 om differentialer

$$\frac{du}{dx} = u'(x) \iff \underbrace{du}_{\text{differential}} = u'(x)\,dx$$

Ex3 $u(x) = x^2$

$du = u'(x)dx = 2x\,dx$.

Ex4 Bestäm arean av område

$D = \{(x,y),\ 0 \le x \le 1,\ 0 \le y \le f(x)\}$

$$f(x) = \frac{x+1}{\sqrt{x^2+2x+3}}$$

Lösning

Vi söker $A(D) = \int_0^1 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx \geq 0$

steg 1 Finn $F(x)$

$$\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = F(x) + C$$

$$\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \left[\begin{array}{l} t = x^2+2x+c \\ \frac{dt}{dx} = 2(x+1) \iff \\ dt = 2(x+1)\,dx \\ \frac{dt}{2} = (x+1)\,dx \end{array}\right.$$

$$= \int \frac{\frac{1}{2}dt}{\sqrt{t}} = \frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt =$$

$$= \frac{1}{2}\int t^{-1/2}\,dt = \left[\int t^{\alpha}dt = \frac{t^{\alpha+1}}{\alpha+1} + c,\ \alpha \neq -1\right]$$

$$= \frac{1}{2}\,\frac{t^{-1/2+1}}{-1/2+1} + 1 = \frac{1}{\cancel{2}}\,\frac{t^{1/2}}{\cancel{\frac{1}{2}}} + c$$

$$\int \frac{\frac{1}{2}\,dt}{\sqrt{t}} = \sqrt{t} + C$$

$$\therefore \int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \sqrt{x^2+2x+3} + C$$

$$\therefore \int_0^1 \frac{x+1}{\sqrt{x^2+2x+3}}\,dx = \left[\sqrt{x^2+2x+3}\right]_0^1 = \ldots$$

$$= \sqrt{1^2+2\cdot 1+3} - \sqrt{0+3} = \sqrt{6} - \sqrt{3} > 0$$

<u>Svar</u> $A(D) = \sqrt{6} - \sqrt{3} > 0$

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Ex 5 (Subst + part. int)

Bestäm $\int_1^4 e^{\sqrt{x}}\,dx = A(D)$

$D = \{(x,y) : 1 \le x \le 4,\ 0 \le y \le e^{\sqrt{x}}\}$

Lösning

steg 1

$$\int e^{\sqrt{x}}\,dx = \left[\begin{array}{l} t = \sqrt{x} = x^{1/2} \\ t^2 = x \Rightarrow \frac{dx}{dt} = 2t \\ \Leftrightarrow dx = 2t\,dt \end{array}\right]$$

$$= \int e^t\, 2t\,dt = 2\int t e^t\,dt$$

= part int $\int u'v\,dt = uv - \int uv'\,dt$

$$= \left[\begin{array}{ll} \text{Val } & u' = e^t \Rightarrow u = e^t \\ & v = t \Rightarrow v' = 1 \end{array}\right]$$

$$\therefore 2\int te^t\,dt = 2\left\{e^t\cdot t - \underbrace{\int e^t \cdot 1\,dt}_{e^t + C}\right\}$$

$$= 2\{e^t t - e^t + C\}$$

Vi måste gå tillbaka till variabel x, där: $t = \sqrt{x}$

$$\int e^{\sqrt{x}}\,dx = 2\{e^{\sqrt{x}}\sqrt{x} - e^{\sqrt{x}} + C\}$$

$$\therefore \int_1^4 e^{\sqrt{x}}\,dx = 2\Big[e^{\sqrt{x}}(\sqrt{x} - 1)\Big]_{x=1}^{x=4}$$

$$= 2\Big(e^{\sqrt{4}}(\sqrt{4} - 1) - \underbrace{e^{\sqrt{1}}(\underbrace{\sqrt{1}-1}_{=0})}_{=0}\Big)$$

$$= 2e^2(2-1)$$

$$= 2e^2 \quad \therefore$$

$$\boxed{\int_1^4 e^{\sqrt{x}}\,dx = 2e^2 > 0}$$

svaret

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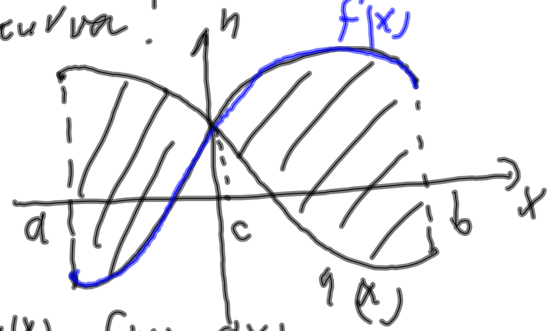
Att beräkna $\int_a^b |u(x)|\,dx$

dvs att beräkna arean mellan två kurvor:



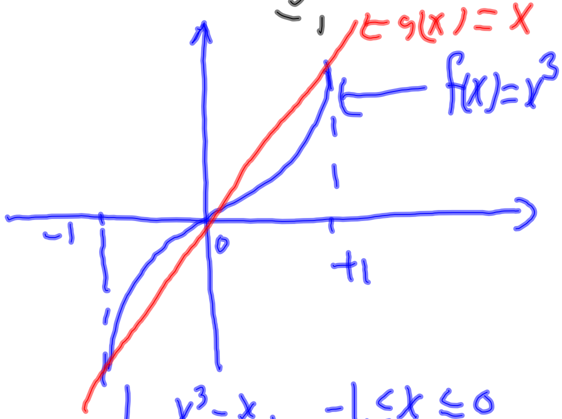
med $u(x) = f(x) - g(x)$

$$\int_a^b |u(x)|\,dx = \int_a^b |f(x)-g(x)|\,dx =$$

$$= \int_a^c -\left[f(x)-g(x)\right]dx + \int_c^b (f(x)-g(x))\,dx$$

<u>Ex 6</u> Beräkna

$$\int_{-1}^{1} |x^3 - x|\,dx \neq \int_{-1}^{1} (x^3 - x)\,dx$$



$$|x^3 - x| = \begin{cases} x^3 - x, & -1 \le x \le 0 \\ -(x^3 - x), & 0 \le x \le 1 \end{cases}$$

$$\int_{-1}^{1} |x^3 - x|\,dx = \int_{-1}^{0} (x^3 - x)\,dx + \int_0^1 -(x^3 - x)\,dx$$

Alt. <u>gör en teckentabell</u>

$$|x^3-x|=0 \iff x^3-x=0$$

$$\iff x(\overbrace{x^2-1}^{(x-1)(x+1)})=0 \implies x=-1,0,1$$

x		-1		0		1	
x	$-$		$-$	0	$+$		$+$
$x-1$	$-$		$-$		$-$	0	$+$
$x+1$	$-$	0	$+$		$+$		$+$
$\|x^3-x\|$	$-$	0	$+$	0	$-$	0	$+$
			(x^3-x)		$-(x^3-x)$		

så här ser ut $|x^3-x|$ i olika intervall

Rep $|x-a| = \begin{cases} x-a, & x \geq a \\ -(x-a), & x \leq a \end{cases}$

$$\int_{-1}^{1} |x^3-x|\,dx = \int_{-1}^{0} (x^3-x)\,dx + \int_{0}^{1} -(x^3-x)\,dx$$

$$= \dots = 1/2$$

Här man använder

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

Ex 7 $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx = \left[\begin{array}{l} t = 1+\sin x \\ \frac{dt}{dx} = \cos x \\ dt = \cos x\,dx \end{array}\right]$

$$= \int \frac{1}{t}\,dt = \ln|t| + C = \ln|1+\sin x| + C$$

Här man använder

$$\int \frac{u'(x)}{1+u^2(x)}\,dx = \arctan(u(x)) + C$$

Ex 8 $\displaystyle\int \frac{\cos x}{1+\sin^2 x}\,dx = \left[\begin{array}{l} t = \sin x \\ \frac{dt}{dx} = \cos x \\ dt = \cos x\,dx \end{array}\right]$

$$= \int \frac{1}{1+t^2}\,dt = \arctan(t) + C$$

$$= \arctan(\sin x) + C$$

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Hur man använde trigonometrin

$$\cos^2 x + \sin^2 x = 1$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$= [\cos^2 x + \sin^2 x = 1]$$
$$= \cos^2 x - (1 - \cos^2 x)$$
$$= 2\cos^2 x - 1$$
$$= (1 - \sin^2 x) - \sin^2 x$$
$$= 1 - 2\sin^2 x$$

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

Exg

$$\int \cos^2 x \, dx = \int \frac{1 + \cos(2x)}{2} dx$$
$$= \int \frac{1}{2} dx + \frac{1}{2} \int \cos(2x) dx$$
$$= \frac{x}{2} + \frac{1}{2} \frac{\sin(2x)}{2} + C$$

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} dx$$
$$= \frac{x}{2} - \frac{1}{2} \frac{\sin(2x)}{2} + C$$

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