A BRIEF INTRODUCTION TO UNIFORM CONVERGENCE

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1. Questions and examples

In the study of Fourier series, several questions arise naturally, such as:

- are there conditions on c_n, $n \in \mathbb{Z}$, which ensure that

$$\sum_{n=-\infty}^{\infty} c_n e^{int} \tag{1}$$

  converges for all t, and, similarly, are there conditions on a_n, $n = 0, 1, \dots,$ and b_n, $n = 1, 2, \dots,$ which ensure that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{2}$$

  converges for all t?
- are there conditions which ensure that (1) and (2) define continuous functions, continuously differentiable functions etc.?
- when is the operation

$$\frac{d}{dt} \sum_{n=-\infty}^{\infty} c_n e^{int} = \sum_{n=-\infty}^{\infty} \frac{d}{dt}(c_n e^{int}) = \sum_{n=-\infty}^{\infty} in c_n e^{int}$$

  (and the similar operation for (2)) allowed?
- when is the operation

$$\int_a^b \left(\sum_{n=-\infty}^{\infty} c_n e^{int} \right) dt = \sum_{n=-\infty}^{\infty} \int_a^b c_n e^{int} dt$$

  (and the similar operation for (2)) allowed?

In order to address these questions, let us consider them in somewhat greater generality. Say that we have a sequence of (complex or real valued) functions f_n, $n = 1, 2, \dots$ or $n = 0, 1, \dots$ (the case $n \in \mathbb{Z}$ can in practice be reduced to two sequences of this form). We are interested in the sum

$$\sum_{n=1}^{\infty} f_n.$$

In order to illustrate that the answers to the above questions are not quite as straightforward as might be expected, let us give an example.

Example 1.1. Let

$$f_n(t) = \frac{t^2}{(1+t^2)^n}$$

for $n = 0, 1, \dots$. Compute

$$s(t) = \sum_{n=0}^{\infty} f_n(t).$$

Solution: If $t = 0$, then $f_n(t) = 0$ for all n. In other words, $s(0) = 0$. For $t \neq 0$, we have

$$s(t) = \sum_{n=0}^{\infty} f_n(t) = \sum_{n=0}^{\infty} \frac{t^2}{(1+t^2)^n} = t^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+t^2}\right)^n.$$

The sum appearing on the right hand side is the sum of a geometric series. In fact, let

$$\alpha = \frac{1}{1+t^2}.$$

Note that, since $t \neq 0$, we have $0 < \alpha < 1$. Compute

$$\sum_{n=0}^{\infty} \left(\frac{1}{1+t^2}\right)^n = \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} = \frac{1}{1-\frac{1}{1+t^2}} = \frac{1+t^2}{1+t^2-1} = \frac{1+t^2}{t^2}.$$

Thus, for $t \neq 0$,

$$s(t) = t^2 \sum_{n=0}^{\infty} \left(\frac{1}{1+t^2}\right)^n = t^2 \frac{1+t^2}{t^2} = 1 + t^2.$$

To conclude

$$s(t) = \begin{cases} 0 & t = 0, \\ 1+t^2 & t \neq 0. \end{cases}$$

In other words, s is *not* a continuous function, since $s(0) = 0$ and $s(t) > 1$ for $t \neq 0$. On the other hand, all the functions f_n can be differentiated an arbitrary number of times. To conclude: even though the sum

$$\sum_{n=0}^{\infty} f_n(t)$$

exists for all t, and even though the partial sums

$$\sum_{n=0}^{N} f_n$$

are continuous (they are in fact k times continuously differentiable for any positive integer k), the sum itself is not even continuous.

Exercise 1.2. Compute

$$s(t) = \sum_{n=0}^{\infty} \frac{t}{(1+t^2)^n}$$

and sketch the graph.

Exercise 1.3. Compute

$$s(t) = \sum_{n=0}^{\infty} \frac{t^3}{(1+t^2)^n}$$

and sketch the graph.

It is possible to construct examples which illustrate that changing the order of integration and summation is also problematic etc. We refer the reader interested in more examples to Walter Rudin's book *Principles of Mathematical Analysis*, Chapter 7.

2. Pointwise and uniform convergence

The reason we obtain the behaviour exhibited in Example 1.1 is that the sum converges *pointwise* but not *uniformly*. In the present section we wish to clarify the meaning of these concepts.

Since we are interested in Fourier series, we are interested in sums of the form

$$s(t) = \sum_{n=1}^{\infty} f_n(t).$$

However, in the various definitions we shall carry out, it is convenient to note that s can be considered to be a limit of a sequence of functions as opposed to a sum of a sequence of functions. In fact, let

$$s_n(t) = \sum_{k=1}^{n} f_k(t).$$

Then

$$s(t) = \lim_{n\to\infty} s_n(t).$$

Let us from now on consider a sequence of functions s_n, $n = 1, 2, \ldots$, and consider the limit of this sequence as $n \to \infty$. Let us define what is meant by *pointwise convergence*.

Definition 2.1. Let s_n, $n = 1, 2, \ldots$, be a sequence of functions defined on an interval I. If the sequence of numbers $s_n(t)$, $n = 1, 2, \ldots$, converges for every $t \in I$, then the sequence of functions s_n, $n = 1, 2, \ldots$, is said to *converge pointwise* on I.

Remark 2.2. If a sequence of functions s_n, $n = 1, 2, \ldots$, converges pointwise on I, then we can define the function s on I by

$$s(t) = \lim_{n\to\infty} s_n(t),$$

and we shall say that the sequence s_n, $n = 1, 2, \ldots$, converges to s *pointwise* on I.

Example 2.3. One sequence which converges pointwise is s_n, $n = 1, 2, \ldots$, where

$$s_n(t) = \sum_{k=0}^{n} \frac{t^2}{(1+t^2)^k};$$

in Example 1.1 we proved that this sequence converges pointwise on $\mathbb{R}$.

Example 2.4. Let

$$s_n(t) = t^n.$$

Then the sequence s_n, $n = 1, 2, \ldots$, converges pointwise on $I = [0, 1]$, since

$$s_n(t) = t^n \to 0$$

if $0 \le t < 1$ and $s_n(1) = 1 \to 1$. If we let

$$s(t) = \lim_{n\to\infty} s_n(t),$$

we thus have

$$s(t) = \begin{cases} 0 & t \in [0,1), \\ 1 & t = 1. \end{cases}$$

In other words, the function s has one point of discontinuity, namely $t = 1$.

Exercise 2.5. Construct a sequence of functions s_n, $n = 1, 2, \ldots$, on $I = [0,1]$ such that

- the functions s_n are all continuous,
- the sequence of functions s_n, $n = 1, 2, \ldots$, converges pointwise to a function s on I,
- s has two points of discontinuity in I.

Can you construct a sequence as in the above exercise such that s has three, four etc. points of discontinuity?

Exercise 2.6. (This exercise is difficult). Construct a sequence of functions s_n, $n = 1, 2, \ldots$, on $I = [0,1]$ such that

- the functions s_n are all continuous,
- the sequence of functions s_n, $n = 1, 2, \ldots$, converges pointwise to a function s on I,
- s has an infinite number of points of discontinuity in I.

The concept of pointwise convergence should be contrasted with the concept of uniform convergence.

Definition 2.7. Let s_n, $n = 1, 2, \ldots$, be a sequence of functions defined on an interval I. Then the sequence s_n, $n = 1, 2, \ldots$, is said to *converge uniformly* on I to a function s if, for every $\epsilon > 0$, there is an N such that $n \geq N$ implies that

$$|s_n(t) - s(t)| < \epsilon$$

for all $t \in I$.

Remark 2.8. If the sequence s_n, $n = 1, 2, \ldots$, converges uniformly on I to a function s, it converges to s pointwise on I (*prove this*).

Let us demonstrate that the sequence considered in Example 2.4 does not converge uniformly. In order to do so, let us assume that the sequence s_n, $n = 1, 2, \ldots$, converges uniformly on $I = [0,1]$ to a function S (if we can deduce a contradiction from this assumption, we are allowed to conclude that the convergence is not uniform). Due to Remark 2.8, the function S has to coincide with the function s appearing in Example 2.4. Due to the definition of uniform convergence, we are allowed to first fix $\epsilon > 0$ (let us assume $\epsilon < 1$). Given this ϵ, there is then an N such that for $n \geq N$, we have

$$|s_n(t) - s(t)| < \epsilon$$

for all $t \in I$. In the case considered in Example 2.4, $s(t) = 0$ for $t \in [0,1)$. We thus have

$$|s_n(t)| < \epsilon$$

for every $t \in [0,1)$. Since the functions s_n are continuous, this means that $|s_n(1)| \leq \epsilon$, but we know that $s_n(1) = 1$. Since $\epsilon < 1$, we have a contradiction. Thus the convergence is not uniform.

Exercise 2.9. Draw a picture to illustrate the above argument.

In order to make the distinction between pointwise and uniform convergence clearer, let us write down the relevant questions to ask in order to check whether one has pointwise or uniform convergence.

Pointwise convergence: first fix a $t \in I$ and then ask if, for every $\epsilon > 0$, there is an N such that for $n \geq N$, $|s_n(t) - s(t)| < \epsilon$ (here N depends on ϵ *and* t).

Uniform convergence: ask if, for every $\epsilon > 0$, there is an N such that for $n \geq N$, $|s_n(t) - s(t)| < \epsilon$ *for all* t (here N *only* depends on ϵ).

Let us, finally, define uniform convergence explicitly for a series.

Definition 2.10. Let f_k, $k = 1, 2, \ldots$, be a sequence of functions defined on an interval I. Then the sum

$$\sum_{k=1}^{\infty} f_k$$

is said to *converge uniformly* on I to s if the the partial sums s_n, $n = 1, 2, \ldots$, where

$$s_n = \sum_{k=1}^{n} f_k,$$

converge uniformly on I to s.

3. Consequences of uniform convergence

The question remains: what is the use of uniform convergence? Let us quote a few theorems that we shall need.

Let us start with a criterion which ensures that the sum is continuous.

Theorem 3.1. *Let f_k, $k = 1, 2, \ldots$, be a sequence of continuous functions defined on an interval I. Assume that the sum*

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly on I to s. Then s is continuous on I.

As was noted in the introduction, it is of interest to find a criterion which allows us to change the order of integration and summation.

Theorem 3.2. *Let f_k, $k = 1, 2, \ldots$, be a sequence of Riemann integrable functions on an interval $I = [a, b]$ (where $-\infty < a < b < \infty$). Assume that the sum*

$$\sum_{k=1}^{\infty} f_k$$

converges uniformly on I to s. Then s is Riemann integrable on I and

$$\int_a^b s(t)dt = \int_a^b \sum_{k=1}^{\infty} f_k(t)dt = \sum_{k=1}^{\infty} \int_a^b f_k(t)dt.$$

Finally, let us address the issue of differentiation.

Theorem 3.3. *Let f_k, $k = 1, 2, \dots$, be a sequence of continuous and continuously differentiable functions defined on an interval I. Assume that the sums*

$$\sum_{k=1}^{\infty} f_k, \quad \sum_{k=1}^{\infty} f_k'$$

converge uniformly on I to s and s_1 respectively. Then s is continuous and continuously differentiable on I, and

$$s' = \left(\sum_{k=1}^{\infty} f_k \right)' = \sum_{k=1}^{\infty} f_k'$$

on I.

We refer the reader interested in a proof of these statements to Walter Rudin's book *Principles of Mathematical Analysis*, Chapter 7.

4. Weierstrass M-test

None of the theorems mentioned in the previous section are very useful unless we have a good criterion which ensures that a series converges uniformly. Such a criterion is provided by the Weierstrass M-test.

Theorem 4.1 (Weierstrass M-test)**.** *Let f_k, $k = 1, 2, \dots$, be a sequence of functions defined on an interval I and assume that there are numbers M_k, $k = 1, 2, \dots$, such that*

$$|f_k(t)| \leq M_k,$$

for $t \in I$ and $k = 1, 2, \dots$, and such that

$$\sum_{k=1}^{\infty} M_k$$

is convergent. Then the sum

$$s(t) = \sum_{k=1}^{\infty} f_k(t)$$

is well defined for all $t \in I$ and $\sum_{k=1}^{\infty} f_k$ converges uniformly on I to s.

Example 4.2. Consider

$$\sum_{k=1}^{\infty} \frac{1}{k^2+1} \sin kt.$$

We wish to use the above results to demonstrate that this sum defines a continuous function. We have

$$f_k(t) = \frac{1}{k^2+1} \sin kt,$$

$k = 1, 2, \dots$. In order to be allowed to apply the Weierstrass M-test, we wish to estimate $|f_k(t)|$. We have

$$|f_k(t)| = \frac{1}{k^2+1} |\sin kt| \leq \frac{1}{k^2+1}.$$

Let us choose

$$M_k = \frac{1}{k^2+1},$$

$k = 1, 2, \ldots$. Then

$$\sum_{k=1}^{\infty} M_k$$

is convergent and Theorem 4.1 applies. Thus the sum

$$s(t) = \sum_{k=1}^{\infty} \frac{1}{k^2 + 1} \sin kt$$

is well defined for all $t \in \mathbb{R}$ and $\sum_{k=1}^{\infty} f_k$ converges uniformly on $\mathbb{R}$ to s. Since the functions f_k are continuous, Theorem 3.1 applies in order to yield the conclusion that s is continuous.

Example 4.3. More generally, if a_n, $n = 0, 1, 2, \ldots$, and b_n, $n = 1, 2, \ldots$, are complex numbers such that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty,$$

then

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

is well defined for all $t \in \mathbb{R}$. Moreover, we obtain uniform convergence due to the Weierstrass M-test and continuity of s due to Theorem 3.1 (when integrating, we can also change the order of summation and integration due to Theorem 3.2).

Example 4.4. By an argument similar to the one presented in Example 4.2, it is possible to prove that if c_n, $n \in \mathbb{Z}$, is a sequence of complex numbers such that

$$\sum_{n=-\infty}^{\infty} |c_n| < \infty,$$

then

$$\sum_{n=-\infty}^{\infty} c_n e^{int},$$

conveges uniformly on $\mathbb{R}$ to a function s. Moreover, s is continuous and, due to Theorem 3.2, we are allowed to change the order of summation and integration in order to conclude that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} s(t) e^{-imt} dt =& \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{n=-\infty}^{\infty} c_n e^{int-imt} \right) dt \\ =& \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int-imt} dt = c_m; \end{aligned}$$

the second equality is justified by Theorem 3.2.

Let us turn to differentiability.

Example 4.5. Consider

$$\sum_{k \in \mathbb{Z}} \frac{1}{(1 + k^2)^2} e^{ikt}.$$

In this case

$$f_k(t) = \frac{1}{(1+k^2)^2}e^{ikt}, \quad f_k'(t) = \frac{ik}{(1+k^2)^2}e^{ikt}.$$

Since

$$|f_k(t)| \leq \frac{1}{(1+k^2)^2}$$

and

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1+k^2)^2} < \infty,$$

we can apply the Weierstrass M-test in order to conclude that

$$\sum_{k\in\mathbb{Z}} f_k$$

conveges uniformly on $\mathbb{R}$ to a function s. Since

$$|f_k'(t)| \leq \frac{|k|}{(1+k^2)^2}$$

and

$$\sum_{k\in\mathbb{Z}} \frac{|k|}{(1+k^2)^2} < \infty,$$

we can also apply the Weierstrass M-test in order to conclude that

$$\sum_{k\in\mathbb{Z}} f_k'$$

conveges uniformly on $\mathbb{R}$ to a function s_1. Due to Theorem 3.3 we thus conclude that s is continuous and continuously differentiable and that

$$s'(t) = \sum_{k\in\mathbb{Z}} \frac{ik}{(1+k^2)^2}e^{ikt}.$$

Example 4.6. By an argument similar to that given in the previous example we have the following conclusion. If $m \geq 1$ is an integer and

$$\sum_{n\in\mathbb{Z}} |n|^m|c_n| < \infty,$$

then

$$s(t) = \sum_{n\in\mathbb{Z}} c_n e^{int}$$

is m-times continuously differentiable and

$$s'(t) = \sum_{n\in\mathbb{Z}} inc_n e^{int}, \dots, s^{(m)}(t) = \sum_{n\in\mathbb{Z}} (in)^m c_n e^{int}.$$

Similarly, if

$$\sum_{n=1}^{\infty} |n|^m(|a_n| + |b_n|) < \infty,$$

then

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

is m times continuously differentiable and we are allowed to differentiate under the summation sign.